

OERTransport: Fundamentals of Math, Physics,
and Statistics for Future Transportation
Professionals

OERTRANSPORT: FUNDAMENTALS OF MATH, PHYSICS, AND STATISTICS FOR FUTURE TRANSPORTATION PROFESSIONALS

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CONTENTS

Preface	ix
About This Project	x
About the Publisher	xi
Mavs Open Press	
<i>About Mavs Open Press</i>	xi
<i>About OER</i>	xi
<i>About Pressbooks</i>	xi
<i>Contact Us</i>	xii
About the Authors	xiii
About the Institution	xiv
Copyright	xv
Acknowledgments	xvi
PART I. MAIN BODY	
Chapter 1: Trigonometry Functions and Geometric Measurements	1
<i>Recognize, Rearrange, and Simplify Trigonometric Functions</i>	1
<i>Identify Properties for Circular, Parabolic, and Spiral Curves</i>	6
<i>Relevance to Transportation Engineering Coursework</i>	7
<i>Media Attributions</i>	9
<i>References</i>	11
Chapter 2: Polynomial, Exponential, and Logarithmic Functions	12
<i>Generate Graphs and Charts from Exponential and Logarithmic Functions</i>	12
<i>Solve the Exponential and Logarithmic Equations</i>	17
<i>Visualize Polynomial Functions on a Graph or Chart</i>	20
<i>Set Up and Solve Polynomial Equations</i>	21
<i>Relevance to Transportation Engineering Coursework</i>	24
<i>Media Attributions</i>	25
<i>References</i>	26

Chapter 3: Systems of Linear Equations	28
<i>Solving Systems of Equations with Two Variables</i>	28
<i>Solving Systems of Equations with Three Variables</i>	31
<i>Systems of Equations in Microsoft Excel</i>	32
<i>Relevance to Transportation Engineering Coursework</i>	34
<i>Media Attributions</i>	35
<i>References</i>	36
Chapter 4: Calculus – Interpretation and Methods for Integration and Differentiation	37
<i>Interpret Derivatives as the Rate of Change</i>	37
<i>Familiarize Oneself with the Rules and Processes (e.g., chain rule, multiplication rule) of Derivation</i>	39
<i>Differentiate Various Functions with Single Variables</i>	42
<i>Integrals of Common Functions</i>	48
<i>Use the Integration Method for Estimating the Area Under a Curve</i>	53
<i>Describe the Process of Integration and Differentiation in Correspondence with Each Other</i>	56
<i>Relevance to Transportation Engineering Coursework</i>	56
<i>Key Takeaways</i>	57
<i>Media Attributions</i>	58
Chapter 5: Motion and Forces	62
<i>Units and Measurements</i>	62
<i>Time, Velocity, Speed, and Acceleration</i>	68
<i>Use Kinematic Equations to Solve for Displacement, Time, Velocity and Acceleration of an Object</i>	75
<i>Free-body Diagrams</i>	79
<i>Estimate the Friction Force and Normal Reaction on a Static Object</i>	82
<i>Forces and Newton’s Laws</i>	85
<i>Describe the Motion of an Object Along a Circular Path</i>	90
<i>Relevance to Transportation Engineering Coursework</i>	110
<i>Media Attributions</i>	111
<i>References</i>	114
Chapter 6: Basic Dynamics and Static Equilibrium	115
<i>Use of Momentum Preservation to Describe the Post-Collision Movement of Objects</i>	115
<i>Balance Forces on a Stationary Object in Equilibrium</i>	132
<i>Relevance to Transportation Engineering Coursework</i>	143
<i>Media Attributions</i>	145
<i>References</i>	147

Chapter 7: Waves and Doppler Effect	148
<i>wave properties</i>	148
<i>Describe the Movement of Sound Waves in the Environment</i>	153
<i>Describe the Doppler Effect in the Context of Waves Emanating from a Moving Source</i>	159
<i>Identify the Causes of Noise from the Transportation System</i>	162
<i>Relevance to Transportation Engineering Coursework</i>	163
<i>Glossary</i>	163
<i>Media Attributions</i>	164
<i>References</i>	166
Chapter 8: Probability: Basic Principles and Distributions	167
<i>Use Basic Counting Techniques to Estimate Probability and Odds</i>	167
<i>Set Up and Work with Distributions for Discrete Random Variables</i>	179
<i>Set Up and Work with Distributions for Continuous Random Variables</i>	200
<i>Relevance to Transportation Engineering Coursework</i>	228
<i>Media Attributions</i>	230
<i>References</i>	234
Chapter 9: Data Analysis - Hypothesis Testing, Estimating Sample Size, and Modeling	235
<i>Central Limit Theorem</i>	235
<i>Use Specific Significance Tests Including, Z-Test, T-Test (one and two samples), Chi-Squared Test</i>	259
<i>Compute and Interpret Simple Linear Regression Between Two Variables</i>	306
<i>Data Analysis Tools (Spreadsheets and Basic Programming)</i>	324
<i>Relevance to Transportation Engineering Coursework</i>	325
<i>Media Attributions</i>	327
<i>References</i>	333
Attributions by Chapter and Derivative Notes	335
<i>Attributions by Chapter</i>	335
<i>Derivative Notes</i>	347
Image Credits	348
Notes	348
Chapter 1	348
Chapter 5	348
Chapter 6	350
Chapter 7	352
Chapter 8	353
Chapter 9	354

PREFACE

This textbook and OER material cover tools and basic knowledge required to master the prerequisite essentials of physics, mathematics, and statistics applied in transportation engineering. To our knowledge, no such textbook currently exists to build the KSTs (Knowledge, Skills, Tools) for college freshmen' remedial courses required by most transportation engineering graduate programs that admit non-engineers (e.g., PHYS 141, MATH 142, STAT 321 at California Polytechnic State University, San Luis Obispo). The textbook modules are intended for students with undergraduate degrees in planning and other less technical fields who are interested in pursuing transportation careers where background engineering knowledge is required.

ABOUT THIS PROJECT

The development of this textbook was made possible by the support of the U.S. Department of Education, which in January 2021 awarded the three-year funded project: **OERTransport: Enabling Transportation Planning Professional Advancement** to the *University of Texas-Arlington (UTA)* in consortium with California Polytechnic State University (Cal Poly) and the University of South Florida (USF).

This open textbook is part of the **OERTransport** collection of six transportation planning OER textbooks and their respective graduate course implementation at each of the three collaborating universities. The creation of each textbook benefitted from an independent industry advisory board's review and the savvy contributions of each institution's OER librarians. For more information on OERTransport: <https://oertransportlab.uta.edu/>

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ABOUT THE INSTITUTION

California Polytechnic State University (Cal Poly) is located in San Luis Obispo on California's Central Coast, about halfway between Los Angeles and San Francisco. Cal Poly was founded in 1901 and has been consistently named the best public, master's-level university in the West by U.S. News & World Report since the late 1990s. It is part of the 23-campus California State University system and is well-known for its learn-by-doing pedagogy.

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CHAPTER 1: TRIGONOMETRY FUNCTIONS AND GEOMETRIC MEASUREMENTS

Basic geometric and trigonometric functions and measurements are important in transportation engineering for the functional design of facilities, including roadways, railroads, and intersection sight distance.

Learning Objectives

- Recognize, rearrange, and simplify trigonometric functions
- Apply properties of triangles to solve for sides and angles
- Identify transportation engineering topics where the properties of triangles and trigonometric functions are used
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter

RECOGNIZE, REARRANGE, AND SIMPLIFY TRIGONOMETRIC FUNCTIONS

This section will explain basic manipulation of trigonometric functions with videos to help your comprehension. Also, short problems to check your understanding are included.

RADIANS & DEGREES



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Check Your Understanding: Radians & Degrees



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SOLVING SIMILAR TRIANGLES



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Check Your Understanding: Solving Similar Triangles



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INTRODUCTION TO THE TRIGONOMETRIC RATIOS



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Check Your Understanding: Introduction to the Trigonometric Ratios



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THE TRIG FUNCTIONS & RIGHT TRIANGLE TRIG RATIOS



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Check Your Understanding: The Trig Functions & Right Triangle Trig Ratios



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SOLVING FOR A SIDE IN A RIGHT TRIANGLE USING THE TRIGONOMETRIC RATIOS



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Check Your Understanding: Solving for a Side in a Right Triangle Using the Trigonometric Ratios



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THE INVERSE TRIGONOMETRIC FUNCTIONS

Introduction to Arcsine



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Introduction to Arctangent



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Introduction to Arccosine



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Check Your Understanding: The Inverse Trigonometric Functions





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LAW OF SINES



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Check Your Understanding: Law of Sines



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LAW OF COSINES



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Check Your Understanding: Law of Cosines



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Check Your Understanding: Recognize, Rearrange, and Simplify Trigonometric Functions Overall



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IDENTIFY PROPERTIES FOR CIRCULAR, PARABOLIC, AND SPIRAL CURVES

This section uses videos to understand the elements of simple circular curves, parabolic curves, and spiral curves.

ELEMENTS OF SIMPLE CIRCULAR CURVES



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ELEMENTS OF PARABOLIC CURVES



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ELEMENTS OF SPIRAL CURVES



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RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section will explain horizontal straight road sections, curves, and spiral transition sections that connect them.

HORIZONTAL PROFILE

Transportation engineers apply the trigonometric functions discussed here in the process of designing roads, and in particular, their horizontal profiles. The horizontal profile describes the horizontal straight road sections, curves, and spiral transition sections (see Figure 1) that connect them. The curves' geometric design elements depend on principles discussed in the following sections of this chapter: "Radians & Degrees", "Solving Similar Triangles", "Introduction to the Trigonometric Ratios", "The Trig Functions & Right Triangle Trig Ratios", "Solving for a Side in a Right Triangle Using the Trigonometric Ratios", and "The Inverse Trigonometric Functions". For details of the design process (discussed in the Fundamentals of Transportation Engineering course) please review the above section titled "Elements of Simple Circular Curves".



Figure 1: Horizontal profile (top view) of a road with significant horizontal curvature.

STOPPING SIGHT DISTANCE ON HORIZONTAL CURVES

The stopping sight distance (SSD) is the minimum distance a driver needs to perceive an unusual situation, such as a deer crossing the road, and stop to avoid a collision. Along horizontal curves, the driver's line of sight may be obstructed by trees or other roadside objects (see Figure 2). The formula for computing the minimum required SSD on horizontal curves near sight obstructions is derived using the principles discussed in the following sections: "Radians & Degrees", "Solving Similar Triangles", "Introduction to the Trigonometric Ratios", "The Trig Functions & Right Triangle Trig Ratios", "Solving for a Side in a Right Triangle Using the Trigonometric Ratios", and "The Inverse Trigonometric Functions".



Figure 2: Horizontal curve near trees.

SIGHT TRIANGLES AT INTERSECTIONS

When approaching an intersection, drivers should be able to see it from a distance to be able to stop in time before colliding with other approaching vehicles, pedestrians, and bicyclists. This distance is computed by first constructing "sight triangles" similar to the one shown in Figure 3. The sight triangle computations are made using the principle of similar triangles (see the section titled "Solving Similar Triangles" above).

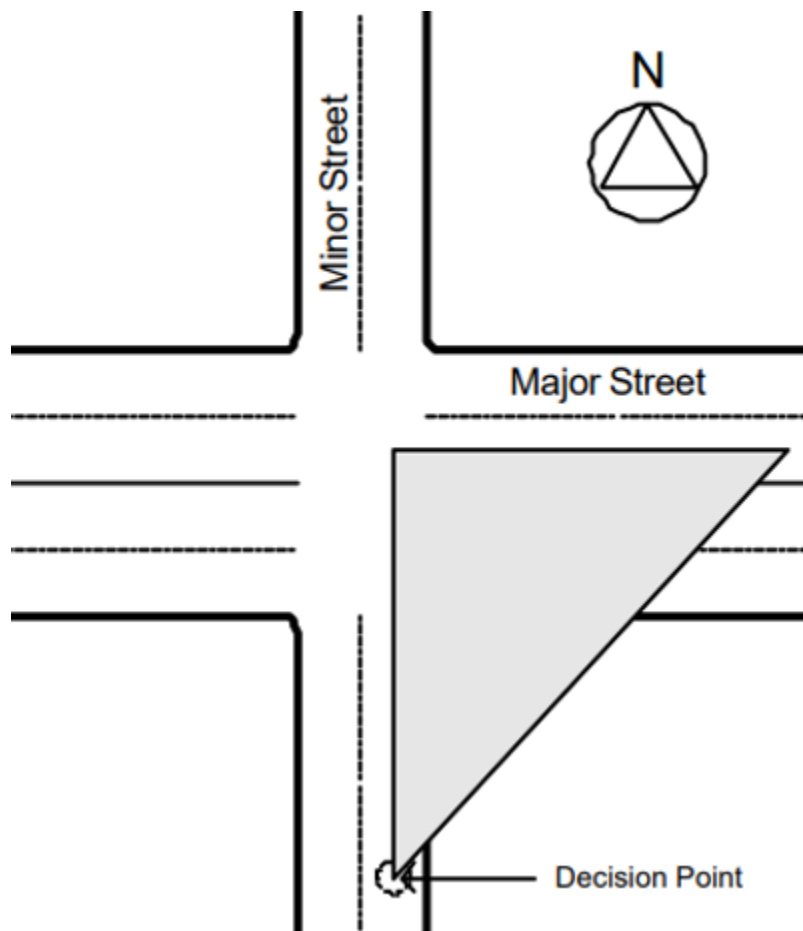


Figure 3: Sample intersection sight triangle.

Key Takeaways

- Transportation engineers apply trigonometric functions to design roads and railroad tracks, particularly their horizontal profiles. Knowledge of these functions is critical to ensuring that on segments with horizontal curvature (or bend around the corner), road users have adequate sight distance available to negotiate the curve on the travel way safely.
- The knowledge of trigonometry is also critical to solving intersection sight triangles and estimating Intersection Sight Distance (ISD) required to safely navigate the intersections.

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- Farid, A. (2022). Highway Design. Personal Collection of Ahmed Farid, California Polytechnic State University, San Luis Obispo, CA.
- Farid, A. (2022). Considerations for Intersections. Personal Collection of Ahmed Farid, California Polytechnic State University, San Luis Obispo, CA.

CHAPTER 2: POLYNOMIAL, EXPONENTIAL, AND LOGARITHMIC FUNCTIONS

Polynomial, exponential, and logarithmic functions have several applications in transport engineering. The exponential functions are used to model economic and population growth and to estimate the compound interest formula. Simplifying polynomial expressions and solving corresponding equations is helpful in the design of transportation facilities and understanding the relationship(s) between flow, speed, and density on roadway segments. Logarithmic functions are also sometimes used to model the relationship between traffic speed and density on a roadway segment, where the speed decreases as the density increases. Understanding the properties of these functions also provide a foundation to learn differential and integral calculus for these functions (See Chapter 4).

Learning Objectives

- Generate graphs and charts from exponential and logarithmic functions.
- Solve the exponential and logarithmic equations.
- Visualize polynomial functions on a graph or chart.
- Set up and solve polynomial equations.
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter

GENERATE GRAPHS AND CHARTS FROM EXPONENTIAL AND LOGARITHMIC FUNCTIONS

This section will explain exponential growth functions, exponential and logarithmic functions, exponential graphs, graphing exponential growth and decay, and introduce you to logarithms with videos to help your comprehension. Also, short problems to check your understanding are included.

INTRODUCTION TO EXPONENTIAL FUNCTIONS





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Check Your Understanding: Exponential Functions



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EXPONENTIAL FUNCTION GRAPH



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GRAPHING EXPONENTIAL FUNCTIONS



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Check Your Understanding: Graphing Exponential Functions





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GRAPHS OF EXPONENTIAL GROWTH



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Check Your Understanding: Graphs of Exponential Growth



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GRAPHING EXPONENTIAL GROWTH AND DECAY



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Check Your Understanding: Graphing Exponential Growth and Decay



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ANALYZING GRAPHS OF EXPONENTIAL FUNCTIONS



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Check Your Understanding: Analyzing Graphs of Exponential Functions



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ANALYZING TABLES OF EXPONENTIAL FUNCTIONS



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Check Your Understanding: Analyzing Tables of Exponential Functions



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INTRODUCTION TO LOGARITHMS



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Check Your Understanding: Logarithms



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GRAPHS OF LOGARITHMIC FUNCTIONS



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Check Your Understanding: Graphs of Logarithmic Functions



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RELATIONSHIP BETWEEN EXPONENTIALS AND LOGARITHMS: GRAPHS



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RELATIONSHIP BETWEEN EXPONENTIALS AND LOGARITHMS: TABLES



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Check Your Understanding: Relationship Between Exponentials and Logarithms



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<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=577#h5p-137>

SOLVE THE EXPONENTIAL AND LOGARITHMIC EQUATIONS

This section will explain how to solve exponential equations using exponent properties in both basic and advanced problems with videos to help your comprehension. Also, short problems to check your understanding are included.

SOLVING EXPONENTIAL EQUATIONS USING EXPONENT PROPERTIES



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Check Your Understanding: Solving Exponential Equations Using Exponent Properties



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SOLVING EXPONENTIAL EQUATIONS USING EXPONENT PROPERTIES (ADVANCED)



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Check Your Understanding: Solving Exponential Equations Using Exponent Properties (Advanced)



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INTRODUCTION TO LOGARITHM PROPERTIES

Please read this link on Introduction to Logarithm Properties.

Check Your Understanding: Introduction to Logarithm Properties



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<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=577#h5p-140>

JUSTIFYING THE LOGARITHM PROPERTIES

Please read this link on Justifying the Logarithm Properties.

LOGARITHM CHANGE OF BASE RULE INTRODUCTION

Please read this link on Logarithm Change of Base Rule Introduction

Check Your Understanding: Justifying the Logarithm Properties & Logarithm Change of Base Rule



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LOGARITHMIC EQUATIONS: VARIABLE IN THE ARGUMENT



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Check Your Understanding: Logarithmic Equations: Variable in the Argument



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LOGARITHMIC EQUATIONS: VARIABLE IN THE BASE



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Check Your Understanding: Logarithmic Equations: Variable in the Base



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VISUALIZE POLYNOMIAL FUNCTIONS ON A GRAPH OR CHART

This section will explain how to graph polynomial functions with videos to help your comprehension. Also, short problems to check your understanding are included.

GRAPHING POLYNOMIAL FUNCTIONS



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GRAPHING LAB – QUADRATICS AND POLYNOMIALS

If you'd like extra practice, please perform this lab.

GRAPHS OF POLYNOMIALS OVERVIEW

Please read this link on Graphs of Polynomials Overview

Check Your Understanding: Graphs of Polynomials



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<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=577#h5p-167>

SET UP AND SOLVE POLYNOMIAL EQUATIONS

This section will explain how to set up and solve polynomials with variables, multiplying binomials with polynomials, and solving polynomial equations by factoring with videos to help your comprehension. Also, short problems to check your understanding are included.

ADDING AND SUBTRACTING POLYNOMIALS

Please read this link on Adding and Subtracting Polynomials

Check Your Understanding: Adding and Subtracting Polynomials



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<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=577#h5p-144>

ADDING AND SUBTRACTING POLYNOMIALS WITH TWO VARIABLES

Please read this link on Adding and Subtracting Polynomials with Two Variables

Check Your Understanding: Adding and Subtracting Polynomials with Two Variables



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<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=577#h5p-145>

MULTIPLYING BINOMIALS BY POLYNOMIALS

Please read this link on Multiplying Binomials by Polynomials

Check Your Understanding: Multiplying Binomials by Polynomials



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<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=577#h5p-146>

SOLVING POLYNOMIAL EQUATIONS BY FACTORING



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FACTORING POLYNOMIALS BY TAKING A COMMON FACTOR

Please read this link on Factoring Polynomials by Taking a Common Factor

Check Your Understanding: Factoring Polynomials by Taking a Common Factor



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<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=577#h5p-147>

DIVIDING POLYNOMIALS: LONG DIVISION



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Check Your Understanding: Dividing Polynomials: Long Division



An interactive H5P element has been excluded from this version of the text. You can view it online here:
<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=577#h5p-148>

SOLVING POLYNOMIAL EQUATIONS BY USING SYNTHETIC SUBSTITUTION



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Check Your Understanding: Solving Polynomial Equations by using Synthetic Substitution



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RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section explains the relevance of engineering economics, traffic assignment, vertical curve design, and traffic flow to transportation engineering coursework.

ENGINEERING ECONOMICS

Engineering economics is the application of economic analysis methods to quantify benefits and costs to assist in the decision-making process. The decisions are related to transportation engineering alternative designs (e.g., constructing a stop-controlled intersection or a roundabout), prioritization of transportation projects, and maintenance budgets, among others. In engineering economics, the time value of money (present value of money and future value of money) and interest rate, also known as the discount rate, need to be considered to provide meaningful Benefit-Cost B/C analysis. The analysis includes the compound interest formula and the exponential growth of money. Exponential functions are also used to model the depreciation of assets. Relevant exponential functions are described in the section titled “Generate Graphs and Charts from Exponential and Logarithmic Functions” of this chapter.

TRAFFIC ASSIGNMENT

In the 4-step travel demand modeling process, traffic assignment is typically the last step, which is used to estimate the number of trips on each route of the roadway network. A critical part of traffic assignment is the Link Performance Function for each route, which defines travel time as a function of the number of users on a route. Travel time increases exponentially as traffic volume increases and is defined based on functional forms discussed in “Generate Graphs and Charts from Exponential and Logarithmic Functions”.

VERTICAL CURVE DESIGN

In highway engineering, one of the fundamental design elements is the vertical profile (i.e., the upgrades, the downgrades, and the curves that connect consecutive roadway sections with different grades, also known as the vertical curves). The vertical curve’s elevation is defined as a quadratic equation, a special type of polynomial function (see “Solve the Exponential and Logarithmic Equations” above).

RELATIONSHIP AMONG TRAFFIC FLOW, SPEED, AND DENSITY

In traffic flow theory for uninterrupted flow facilities (such as freeways), the fundamental

relationship relates speed and density on the segment with the flow rate past a point on the road segment. The decrease in speed as segment density increases may be modeled using linear, polynomial, or logarithmic functions. The nature of speed-density functions may be used to model relationships between the flow rate and density and/or the flow rate and speed. Polynomial and Logarithmic functions discussed in this chapter are critical to this analysis.

Key Takeaways

- Exponential functions are useful for transportation practitioners in applying the compound interest formula and modeling asset depreciation and population growth rate in engineering economics.
- Exponential functions are also used in defining link performance functions in traffic assignment to model travel time as a function of road users on a network route
- Logarithmic functions are sometimes used to define speed as a function of roadway segment density.
- Polynomial functions are used in several applications, including the equation of parabolic curves connecting two consecutive grades on a roadway segment.
- Speed-flow and flow-density relationships on roadway segments are often also defined as polynomial functions.

MEDIA ATTRIBUTIONS

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VIDEOS

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- Video 19: Solving Polynomial Equations by using Synthetic Substitution by Keith Mann is licensed by Creative Commons Attribution 3.0 Unported (CC BY 3.0)

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- Farid, A. (2022). *Transportation Planning 2*. Personal Collection of Ahmed Farid, California Polytechnic State University, San Luis Obispo, CA.
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CHAPTER 3: SYSTEMS OF LINEAR EQUATIONS

The ability to solve systems of multiple equations is a critical component of the introductory transportation course. Systems of equations are helpful in several key transportation analyses, including travel demand estimation (e.g., traffic assignment), design (e.g., roadway alignment), and operations (e.g., speed-density relationships). This chapter highlights how to solve systems of equations with two variables, three variables, and when no solution occurs. It will also explain how to use Microsoft Excel to solve system equations.

Learning Objectives

- Solve a system of equations of up to three variables.
- Implement solving equations up to five variables in Microsoft Excel.
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter

SOLVING SYSTEMS OF EQUATIONS WITH TWO VARIABLES

In this section, you will learn how to setup a system of equations and solve linear equations by reading each description along with watching the videos. Also, short problems to check your understanding are included.

HOW TO SET UP A SYSTEM OF EQUATIONS



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://uta.pressbooks.pub/oert-mpsfundamentals/?p=477#oembed-1>

Check Your Understanding: How to Set Up a System of Equations



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HOW TO SOLVE SYSTEMS OF EQUATIONS GRAPHICALLY



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Check Your Understanding: How to Solve Systems of Equations Graphically



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TESTING A SOLUTION TO A SYSTEM OF EQUATIONS



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Check Your Understanding: Testing a Solution to a System of Equations



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SOLVING SYSTEM OF EQUATIONS WITH SUBSTITUTION



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Check Your Understanding: Solving System of Equations with Substitution



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SOLVING SYSTEMS OF EQUATIONS WITH ELIMINATION



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Check Your Understanding: Solving Systems of Equations with Elimination



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<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=477#h5p-68>

SOLVING SYSTEMS OF EQUATIONS WITH THREE VARIABLES

In this section, you will learn how to set up a system of equations with three variables and solve a system of equations when no solution occurs by watching the videos. Short problems to check your understanding are included.

INTRODUCTION TO LINEAR SYSTEMS WITH THREE VARIABLES



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SOLVING LINEAR SYSTEMS WITH THREE VARIABLES



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Check Your Understanding: Solving Linear Systems with Three Variables



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<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=477#h5p-69>

SOLVING LINEAR SYSTEMS WITH THREE VARIABLES: NO SOLUTION



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Check Your Understanding: Solving Linear Systems with Three Variables: No Solution



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SYSTEMS OF EQUATIONS IN MICROSOFT EXCEL

In this section, you will learn how to set up a system of equations using Microsoft Excel and matrices and how to use the Microsoft Excel solver tool by watching the videos. Short problems to check your understanding are included.

SOLVING SYSTEMS OF EQUATIONS USING MATRICES



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Check Your Understanding: Solving Systems of Equations Using Matrices



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SOLVING SYSTEM OF EQUATIONS USING EXCEL



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Check Your Understanding: Solving System of Equations Using Excel



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HOW TO USE THE SOLVER TOOL IN EXCEL TO SOLVE SYSTEMS OF LINEAR EQUATIONS IN ALGEBRA



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Check Your Understanding: How to Use the Solver Tool in Excel to Solve Systems of Linear Equations in Algebra



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<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=477#h5p-73>

RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section explains the relevance of traffic assignment and operation along with highway design to transportation engineering coursework.

TRAFFIC ASSIGNMENT

In travel demand modeling process, the last step is the traffic assignment for routes, which involves estimating the number of trips made on each route on the roadway network. Two commonly applied traffic assignment approaches, which involve solving a system of equations, are the user equilibrium (UE) and system optimal (SO) assignments. For simple UE problems helpful in conceptual understanding of these approaches, the system of equations is typically linear. The number of variables to solve for depends on the number of routes being considered for traffic assignment. The reader is referred to the above section titled “Solving Systems of Equations with Two Variables” for solving a system of linear equations with two variables and the section titled “Systems of Equations in Microsoft Excel” for solving a system of linear equations with three variables. For solving a system with more than three variables one requires use of computer-based tools which are discussed in “Systems of Equations in Microsoft Excel”.

HIGHWAY DESIGN

Analysis and design of roadway (or railway) elements, particularly horizontal alignment and vertical profiles, often require solving a system of equations. These equations are set up based on the geometric constraints of the design. For example, the roadway elevation at a certain location may be fixed due to an intersecting element. These constraints are used to set up system of simultaneous equations that require the use of methods discussed in the three previous sections in this chapter for solving them.

TRAFFIC OPERATIONS

The relationships between highway speed, lane density (a measure of congestion), and traffic flow (number of vehicles past a point) are governed by a universal relationship of uninterrupted flow, i.e., flow rate is a multiple of speed and density. In addition, different segments are characterized by certain relationships between any of the variable pairs (i.e., speed-density; speed-flow; or flow-density). These relationships help set up equations for key parameters for roadway segments such as capacity flow rate, jam density, and free flow speed.

Key Takeaways

- There are two main approaches to traffic assignment in the travel demand modeling process: user equilibrium (UE) and system optimal (SO). Both of these approaches involve solving a system of equations to determine the number of trips that will be made on each route.
- In highway design equations that help ensure that the final design meets all necessary requirements and constraints need to be solved using the methods described in the chapter.
- In traffic operations, different segments of roadways with uninterrupted flow may have different relationships between speed, flow, and density, and these relationships can be used to set up equations for important parameters such as capacity flow rate, jam density, and free flow speed.

MEDIA ATTRIBUTIONS

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CHAPTER 4: CALCULUS – INTERPRETATION AND METHODS FOR INTEGRATION AND DIFFERENTIATION

Knowing the interpretation of derivatives helps in understanding the origin of formulas for estimating values for roadway elevation along the vertical curve. Understanding derivatives as a rate of change can also help with traffic operations (e.g., capacity as a maximum flow point) and transportation economics (e.g., marginal costs and congestion pricing). Integrals are used to estimate the area under a given curve. This property of integrals is used in estimating earthwork volumes and quantifying aggregate delays at intersections or roadway segments with incidents.

Learning Objectives

- Interpret derivatives as the rate of change.
- Differentiate various functions with single variables.
- Familiarize oneself with the rules and processes (e.g., chain rule, multiplication rule) of derivation.
- Use the integration method for estimating the area under a curve.
- Describe the process of integration and differentiation in correspondence with each other.
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter

INTERPRET DERIVATIVES AS THE RATE OF CHANGE

In this section, you will learn about derivatives, tangent slopes, tangent lines, quotient rule, chain rule and power rule by watching the videos. Also, short problems to check your understanding are included.

INTERPRETING DERIVATIVES



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DERIVATIVE AS SLOPE OF CURVE



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Check Your Understanding: Interpreting Derivatives & Derivative as Slope of Curve



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THE DERIVATIVE AS SLOPE OF TANGENT LINE



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Check Your Understanding: The Derivative as Slope of Tangent Line



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TANGENT SLOPE AS INSTANTANEOUS RATE OF CHANGE



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Check Your Understanding: Tangent Slope as Instantaneous Rate of Change



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APPROXIMATING INSTANTANEOUS RATE OF CHANGE WITH AVERAGE RATE OF CHANGE



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FAMILIARIZE ONESELF WITH THE RULES AND PROCESSES (E.G., CHAIN RULE, MULTIPLICATION RULE) OF DERIVATION

BASIC DERIVATIVE RULES



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BASIC DERIVATIVE RULES (PART 2)



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Check Your Understanding: Basic Derivative Rules



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PRODUCT RULE



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Check Your Understanding: Product Rule



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QUOTIENT RULE



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Check Your Understanding: Quotient Rule



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CHAIN RULE



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Check Your Understanding: Chain Rule



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THE POWER RULE



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Check Your Understanding: The Power Rule



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DERIVATIVES OF TRIGONOMETRIC FUNCTIONS



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Check Your Understanding: Derivatives of Trigonometric Functions



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DIFFERENTIATE VARIOUS FUNCTIONS WITH SINGLE VARIABLES

In this section, you will learn how to differentiate various functions, products, quotients, rational

functions, trigonometric functions, and logarithmic by watching the videos. Also, short problems to check your understanding are included.

DIFFERENTIABILITY AT A POINT: ALGEBRAIC



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Check Your Understanding: Differentiability at a Point; Algebraic



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DIFFERENTIATING POLYNOMIALS



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Check Your Understanding: Differentiating Polynomials



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FRACTIONAL POWERS DIFFERENTIATION



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Check Your Understanding: Fractional Powers Differentiation



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RADICAL FUNCTIONS DIFFERENTIATION INTRODUCTION



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WORKED EXAMPLE



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Check Your Understanding: Radical Functions Differentiation



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DIFFERENTIATING PRODUCTS



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Check Your Understanding: Differentiating Products



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DIFFERENTIATE QUOTIENTS

Please read this link on differentiate quotients.

Check Your Understanding: Differentiate Quotients



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DIFFERENTIATING RATIONAL FUNCTIONS



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Check Your Understanding: Differentiating Rational Functions



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DIFFERENTIATE TRIGONOMETRIC FUNCTIONS



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Check Your Understanding: Differentiate Trigonometric Functions



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DIFFERENTIATE EXPONENTIAL FUNCTIONS



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Check Your Understanding: Differentiate Exponential Functions



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DIFFERENTIATE LOGARITHMIC FUNCTIONS



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Check Your Understanding: Differentiate Logarithmic Functions



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DIFFERENTIATING USING MULTIPLE RULES

Common Derivatives Review

Please read this link on common derivatives.

Derivative Rules Review

Check Your Understanding: Differentiating Using Multiple Rules



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INTEGRALS OF COMMON FUNCTIONS

In this section, you will learn about indefinite integrals of $\sin(x)$, $\cos(x)$, and e^x and definite integrals functions, and U-substitution exponential function by watching the videos. Also, short problems to check your understanding are included.

INDEFINITE INTEGRALS OF COMMON FUNCTIONS

Indefinite Integral of $1/x$



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Check Your Understanding: Indefinite Integral of $1/x$



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Indefinite Integrals of $\sin(x)$, $\cos(x)$, and e^x



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Check Your Understanding: Indefinite Integrals of $\sin(x)$, $\cos(x)$, and e^x



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DEFINITE INTEGRALS OF COMMON FUNCTIONS

Reverse Power Rule



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Check Your Understanding: Reverse Power Rule



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Rational Functions



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Check Your Understanding: Rational Functions



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Radical Functions



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Check Your Understanding: Radical Functions



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Trig Functions



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Check Your Understanding: Trig Functions



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Natural Logs



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Check Your Understanding: Natural Logs



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Absolute Value Functions



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Check Your Understanding: Absolute Value Functions



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Piecewise Functions



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Check Your Understanding: Piecewise Functions



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Integrating with U-Substitution



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U-Substitution: Definite Integral of Exponential Function



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Check Your Understanding: Integrating with U-Substitution



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USE THE INTEGRATION METHOD FOR ESTIMATING THE AREA UNDER A CURVE

In this section, you will learn about integral calculus, definite integrals, and the Riemann calculus technique by watching the videos. Also, short problems to check your understanding are included.

INTRODUCTION TO INTEGRAL CALCULUS



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INTRODUCTION TO DEFINITE INTEGRALS



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INTRODUCTION TO RIEMANN APPROXIMATION



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Riemann Sums Review

Go to this link to read information on Riemann Sums.

Check Your Understanding: Riemann Approximation



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DEFINITE INTEGRAL AS THE LIMIT OF A RIEMANN SUM



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Check Your Understanding: Definite Integral as the Limit of a Riemann Sum



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FINDING AREA UNDER A CURVE USING INTEGRATION



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Check Your Understanding: Finding Area Under a Curve Using Integration



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DESCRIBE THE PROCESS OF INTEGRATION AND DIFFERENTIATION IN CORRESPONDENCE WITH EACH OTHER

THE FUNDAMENTAL THEOREM OF CALCULUS



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Check Your Understanding: The Fundamental Theorem of Calculus



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RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section explains the relevance of designing vertical curves, estimating congestion pricing schedules, traffic operations and roadway capacity, and estimating aggregate delays in transportation engineering coursework.

DESIGN OF VERTICAL CURVES

In roadway design, one grade or slope is connected with the next grade (For example, an upgrade of 3% and a downgrade of 2%) using a parabolic curve defined by a quadratic equation. First-order derivatives of these equations are used to define the highest or lowest point on these curves. If the road is being transitioned from an uphill to a downhill (or vice versa), then the highest point (or lowest point) also represents the point at which the slope is zero. Furthermore, the second derivative of the quadratic equation is a constant, i.e., independent of one's location on the curve. It provides for a smooth transition at a constant rate between the two grades. In this chapter, the sections titled “Interpret Derivatives as the Rate of Change” and “Differentiate Various Functions with Single Variables” provided the details on materials relevant to differentiating polynomials.

ESTIMATING CONGESTION PRICING SCHEDULES

The idea of congestion pricing is to direct roadway users to use transportation network elements (i.e., routes) that would minimize costs (in the form of travel times) for all road users.

Developing a fundamental understanding of this idea requires the first-order differential of the function for the costs incurred by all road users to be set to zero. In this chapter, the sections titled “Interpret Derivatives as the Rate of Change” and “Differentiate Various Functions with Single Variables” provide the details on the relevant materials.

TRAFFIC OPERATIONS AND ROADWAY CAPACITY

Roadway capacity is defined as the maximum flow rate across a point on the roadway segment. It is defined based on the maxima of speed-flow or speed-density relationships. The maxima is found by setting the first derivative of a speed-density (or speed-flow) relationship to zero. In this chapter, the sections titled “Interpret Derivatives as the Rate of Change” and “Differentiate Various Functions with Single Variables” provide the relevant background for this application.

EARTHWORK VOLUMES

Roadway construction often requires earthwork, i.e., the addition or removal of material from the sites. The design process yields the roadway elevation and cross-section along a segment. These point locations are then used to estimate the aggregate earthwork volume using the principles described in the section titled “Use the Integration Method for Estimating the Area Under a Curve” of this chapter.

ESTIMATING AGGREGATE DELAYS

The estimation of aggregate delays requires the estimation of the area under the curve for the input-output diagrams. These diagrams have time on the x-axis and the number of road users entering/exiting a roadway location on the y-axis. The time of entrance for a road user defines the input curve, while the time of exit defines the output curve. The area between the two curves defines the aggregate delay for the roadway entity under consideration (i.e., an intersection or roadway segment). The above section titled “Use the Integration Method for Estimating the Area Under a Curve” provides the details needed for this chapter.

KEY TAKEAWAYS

Key Takeaways

- The interpretation of first-order derivative as a rate of change and definition of maxima or minima of function as points where this derivative is equal to zero is critical to several transport applications. It is used in roadway design for a smooth transition between grades, minimization of total user costs via congestion pricing, and estimating roadway capacity based on speed-flow or flow-density relationships.
- Integration of a function over a defined interval may be interpreted as the area under the curve. This interpretation is used to estimate the volume of earthwork required, which is a common aspect of roadway construction. This information is important for anticipating the resources and equipment needed for planning and budgeting purposes. The same interpretation of integral as area under the

curve is used to estimate the area under the input-output diagrams in traffic operations to estimate aggregate delay on intersections and roadway locations as well

MEDIA ATTRIBUTIONS

Note: All Khan Academy content is available for free at (www.khanacademy.org).

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CHAPTER 5: MOTION AND FORCES

Basic elements of accelerated motion and motion at constant speed help estimate braking distance and safe travel speeds. Concepts of motion along a straight line are also helpful in estimating headways and flow rates on the facilities. Accelerated motion along a circular path provides a conceptual understanding of roadway superelevation (or banking; also see Chapter 6).

This chapter also describes forces acting on static objects. An understanding of friction forces and normal reactions is critical to transportation infrastructure design. The gravity model is useful as a prerequisite to understanding the dynamics of travel demand (trip distribution between traffic analysis zones (TAZs)).

Learning Objectives

- Describe the gravity model
- Estimate the friction force and Normal Reaction on a static object
- Describe the motion of an object in the graphic form through the time-space diagram
- Use kinematic equations to solve for displacement, time, velocity, and acceleration of an object.
- Describe the motion of an object along a circular path
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter

UNITS AND MEASUREMENTS

PHYSICAL QUANTITIES AND UNITS

We define a physical quantity either by specifying how it is measured or by stating how it is calculated from other measurements. For example, we define distance and time by specifying methods for measuring them, whereas we define average speed by stating that it is calculated as distance traveled divided by time of travel.

Measurements of physical quantities are expressed in terms of units, which are standardized values. For example, the length of a race, which is a physical quantity, can be expressed in units of meters (for sprinters) or kilometers (for distance runners). Without standardized units,

it would be extremely difficult for scientists to express and compare measured values in a meaningful way.

There are two major systems of units used in the world: SI units (also known as the metric system) and English units (also known as the customary or imperial system). English units were historically used in nations once ruled by the British Empire and are still widely used in the United States. Virtually every other country in the world now uses SI units as the standard; the metric system is also the standard system agreed upon by scientists and mathematicians. The acronym “SI” is derived from the French *Système International*.

SI UNITS OF TIME, LENGTH, AND MASS

SI units are part of the metric system. Metric systems have the advantage that conversions of units involve base-10 number system. There are 100 centimeters in a meter, 1000 meters in a kilometer, and so on. In nonmetric systems, such as the system of U.S. customary units, the relationships are not as simple—there are 12 inches in a foot, 5280 feet in a mile, and so on.

Length	Mass	Time
Meter (m)	Kilogram (kg)	Second (s)

It is often necessary to convert from one type of unit to another. Let us consider a simple example of how to convert units. Let us say that we want to convert 80 meters (m) to kilometers (km). The first thing to do is to list the units that you have and the units that you want to convert to. In this case, we have units in meters, and we want to convert to kilometers.

Next, we need to determine a conversion factor relating meters to kilometers. A conversion factor is a ratio expressing how many of one unit are equal to another unit. For example, there are 12 inches in 1 foot, 100 centimeters in 1 meter, 60 seconds in 1 minute, and so on. In this case, we know that there are 1,000 meters in 1 kilometer.

Now we can set up our unit conversion. We will write the units that we have and then multiply them by the conversion factor so that the units cancel out, as shown:

$$80 \cancel{\text{m}} \times \frac{1 \text{ km}}{1000 \cancel{\text{m}}} = 0.080 \text{ km.}$$

Note that the “m” unit cancels, leaving only the desired “km” unit. You can use this method to convert between any types of units.

DISPLACEMENT

Position:

To describe the motion of an object, you must first be able to describe its position—where it is at any particular time. More precisely, you need to specify its position relative to a convenient reference frame. Earth is often used as a reference frame, and we often describe the position of

an object as it relates to stationary objects in that reference frame. For example, a rocket launch would be described in terms of the position of the rocket with respect to the Earth as a whole, while a professor's position could be described in terms of where she is in relation to the nearby white board. (See Figure 1). In other cases, we use reference frames that are not stationary but are in motion relative to the Earth. To describe the position of a person in an airplane, for example, we use the airplane, not the Earth, as the reference frame. (See Figure 2).

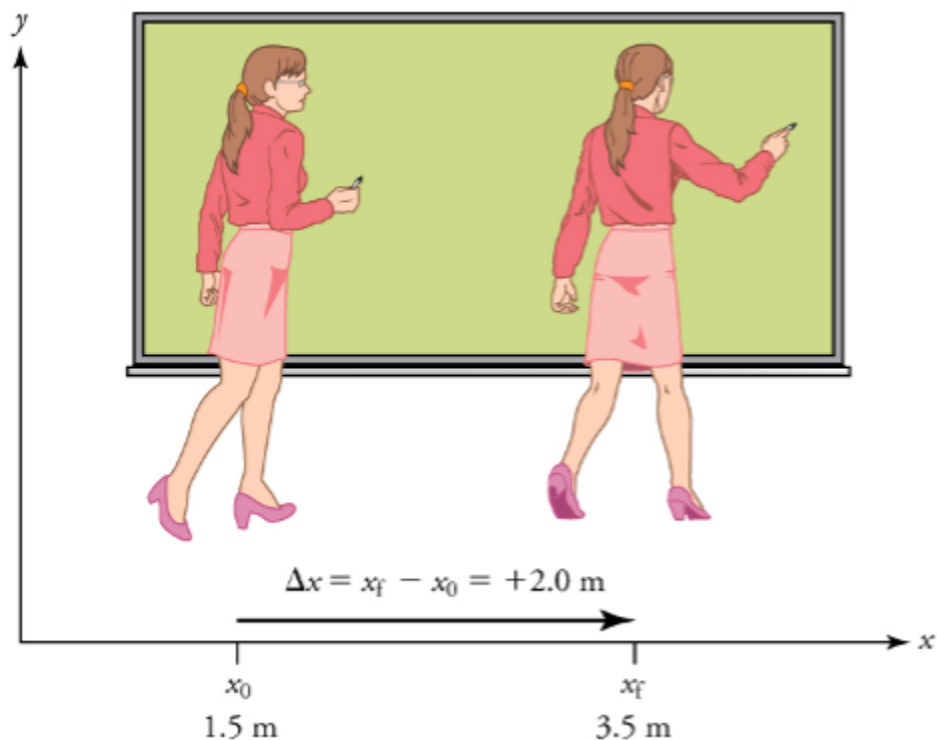


Figure 1: A professor paces left and right while lecturing. Her position relative to the blackboard is given by x . The $+2.0\text{m}$ displacement of the professor relative to the blackboard is represented by an arrow pointing to the right.

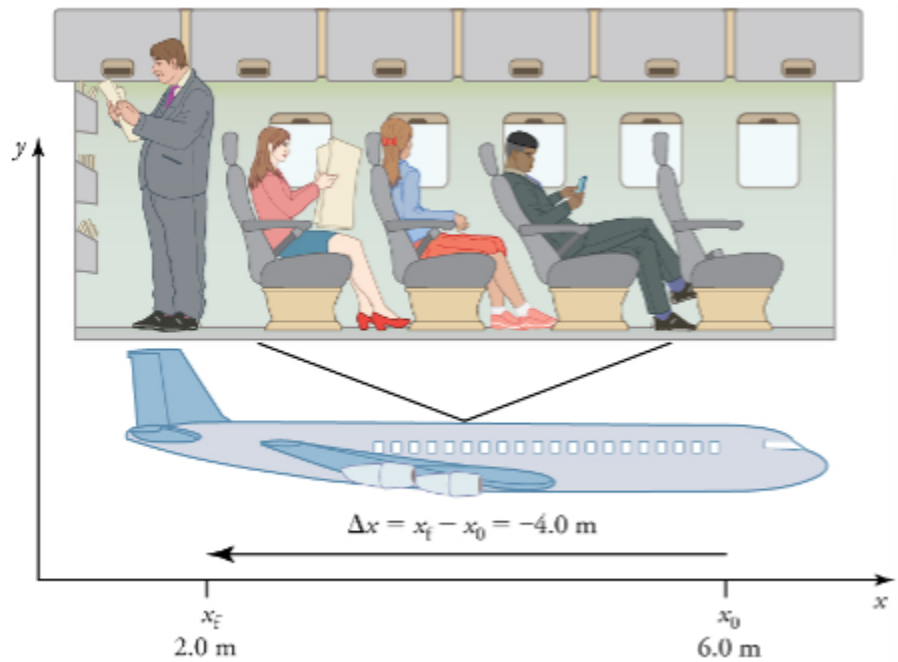


Figure 2: A passenger moves from his seat to the back of the plane. His location relative to the airplane is given by x . The -4 m displacement of the passenger relative to the plane is represented by an arrow toward the rear of the plane. Notice that the arrow representing his displacement is twice as long as the arrow representing the displacement of the professor (he moves twice as far) in Figure 1.

Displacement:

If an object moves relative to a reference frame (for example, if a professor moves to the right relative to a white board or a passenger moves toward the rear of an airplane), then the object's position changes. This change in position is known as **displacement**. The word “displacement” implies that an object has moved or has been displaced.

Displacement is the *change in position* of an object:

$$\Delta x = x_f - x_0$$

Where Δx is displacement, x_f is the final position, and x_0 is the initial position.

The Greek letter Δ (delta) means “change in” whatever quantity follows it; thus, Δx means *change in position*. Always solve for displacement by subtracting initial position x_0 from final position x_f .

Note that the SI unit for displacement is the meter (m), but sometimes kilometers, miles, feet, and other units of length are used. Keep in mind that when units other than the meter are used in a problem, you may need to convert them into meters to complete the calculation.

Note that displacement has a direction as well as a magnitude. The professor's displacement is 2.0 m to the right, and the airline passenger's displacement is 4.0 m toward the rear. In one-dimensional motion, direction can be specified with a plus or minus sign. When you begin a

problem, you should select which direction is positive (usually that will be to the right or up, but you are free to select positive as being any direction). The professor's initial position is $x_0 = 1.5$ m and her final position is $x_f = 3.5$ m. Thus, her displacement is

$$\Delta x = x_f - x_0 = 3.5\text{m} - 1.5\text{m} = +2.0\text{m}$$

In this coordinate system, motion to the right is positive, whereas motion to the left is negative. Similarly, the airplane passenger's initial position is $x_0 = 6.0$ m and his final position is $x_f = 2.0$ m, so his displacement is

$$\Delta x = x_f - x_0 = 2.0\text{ m} - 6.0\text{ m} = -4.0\text{ m}$$

His displacement is negative because his motion is toward the rear of the plane, or in the negative x direction in our coordinate system.

Distance

Although displacement is described in terms of direction, distance is not. Distance is defined to be the magnitude or size of displacement between two positions. Note that the distance between two positions is different from the distance traveled between them. Distance traveled is the total length of the path traveled between two positions. Distance has no direction and, thus, no sign. For example, the distance the professor walks is 2.0 m. The distance the airplane passenger walks is 4.0 m.

Check Your Understanding: Displacement



An interactive H5P element has been excluded from this version of the text. You can view it online here: <https://uta.pressbooks.pub/oert-mpsfundamentals/?p=544#h5p-117>

VECTORS, SCALARS, AND COORDINATE SYSTEMS

What is the difference between distance and displacement? Whereas displacement is defined by both direction and magnitude, distance is defined only by magnitude. Displacement is an example of a vector quantity. Distance is an example of a scalar quantity. A **vector** is any quantity with *both magnitude and direction*. Other examples of vectors include a velocity of 90 km/h east and a force of 500 newtons straight down.

The direction of a vector in one-dimensional motion is given simply by a plus (+) or minus (−) sign. Vectors are represented graphically by arrows. An arrow used to represent a vector has

a length proportional to the vector's magnitude (e.g., the larger the magnitude, the longer the length of the vector) and points in the same direction as the vector.

Some physical quantities, like distance, either have no direction or none is specified. A **scalar** 20°C

is any quantity that has a magnitude, but no direction. For example, a 20°C temperature, the 250 kilocalories (250 Calories) of energy in a candy bar, a 90 km/h speed limit, a person's 1.8 m height, and a distance of 2.0 m are all scalars – quantities with no specified direction. Note, however, that a scalar can be negative, such as -20°C temperature. In this case, the minus sign indicates a point on a scale rather than a direction. Scalars are never represented by arrows.

Coordinate Systems for One-Dimensional Motion

To describe the direction of a vector quantity, you must designate a coordinate system within the reference frame. For one-dimensional motion, this is a simple coordinate system consisting of a one-dimensional coordinate line. In general, when describing horizontal motion, motion to the right is usually considered positive, and motion to the left is considered negative. With vertical motion, motion up is usually positive, and motion down is negative. In some cases, however, as with the jet in Figure 3 below, it can be more convenient to switch the positive and negative directions. For example, if you are analyzing the motion of falling objects, it can be useful to define downwards as the positive direction. If people in a race are running to the left, it is useful to define left as the positive direction. It does not matter as long as the system is clear and consistent. Once you assign a positive direction and start solving a problem, you cannot change it.

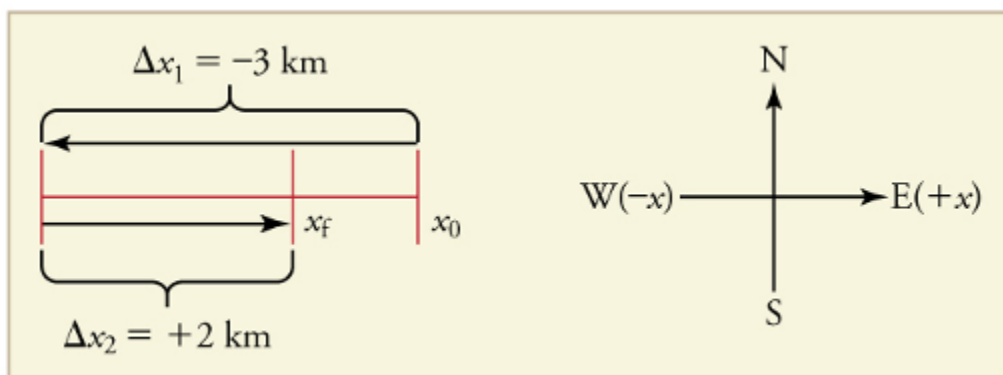


Figure 3: It is usually convenient to consider motion upward or to the right as positive (+) and motion downward or to the left as negative (-)

Check Your Understanding: Vectors, Scalars, and Coordinate Systems



An interactive H5P element has been excluded from this version of the text. You can view it online here:
<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=544#h5p-118>

TIME, VELOCITY, SPEED, AND ACCELERATION

In this section, you will learn about time, velocity, speed and acceleration along with motion graphs and motion diagrams by reading each description. Also, short problems to check your understanding are included.

DESCRIBE THE MOTION OF AN OBJECT IN THE GRAPHIC FORM THROUGH THE TIME-SPACE DIAGRAM

Time, Velocity, and Speed, Acceleration

In physics, the definition of time is simple — time is change, or the interval over which change occurs. It is impossible to know that time has passed unless something changes. The amount of time or change is calibrated by comparison with a standard. The SI unit for time is the second, abbreviated s. We might, for example, observe that a certain pendulum makes one full swing every 0.75 s. We could then use the pendulum to measure time by counting its swings or, of course, by connecting the pendulum to a clock mechanism that registers time on a dial. This allows us to not only measure the amount of time, but also to determine a sequence of events.

How does time relate to motion? We are usually interested in elapsed time for a particular motion, such as how long it takes an airplane passenger to get from his seat to the back of the plane. To find elapsed time, we note the time at the beginning and end of the motion and subtract the two. For example, a lecture may start at 11:00 A.M. and end at 11:50 A.M., so that the elapsed time would be 50 min. Elapsed time Δt is the difference between the ending time and beginning time,

$$\Delta t = t_f - t_0$$

Where Δt is the change in time or elapsed time, t_f is the time at the end of the motion, and t_0 is the time at the beginning of the motion. (As usual, the delta symbol, Δ , means the change in the quantity that follows it.)

Velocity:

Your notion of velocity is probably the same as its scientific definition. You know that if you have a large displacement in a small amount of time you have a large velocity, and

that velocity has units of distance divided by time, such as miles per hour or kilometers per hour.

Average velocity is *displacement (change in position divided by the time of travel,*

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_0}{t_f - t_0}$$

Where \bar{v} is the *average* (indicated by the bar over the v) velocity, Δx is the change in position (or displacement), and x_f and x_0 are the final and beginning positions at times t_f and t_0 , respectively. If the starting time t_0 is taken to be zero, the average velocity is simply

$$\bar{v} = \frac{\Delta x}{t}$$

Notice that this definition indicates that velocity is a vector because displacement is a vector. It has both magnitude and direction. The SI unit for velocity is meters per second or m/s, but many other units, such as km/h, mi/h (also written as mph), and cm/s, are in common use. Suppose, for example, an airplane passenger took 5 seconds to move -4 m (the minus sign indicates that displacement is toward the back of the plane). His average velocity would be

$$\bar{v} = \frac{\Delta x}{t} = \frac{-4 \text{ m}}{5 \text{ s}} = -0.8 \text{ m/s}$$

The minus sign indicates the average velocity is also toward the rear of the plane.

The average velocity of an object does not tell us anything about what happens to it between the starting point and ending point, however. For example, we cannot tell from average velocity whether the airplane passenger stops momentarily or backs up before he goes to the back of the plane. To get more details, we must consider smaller segments of the trip over smaller time intervals.

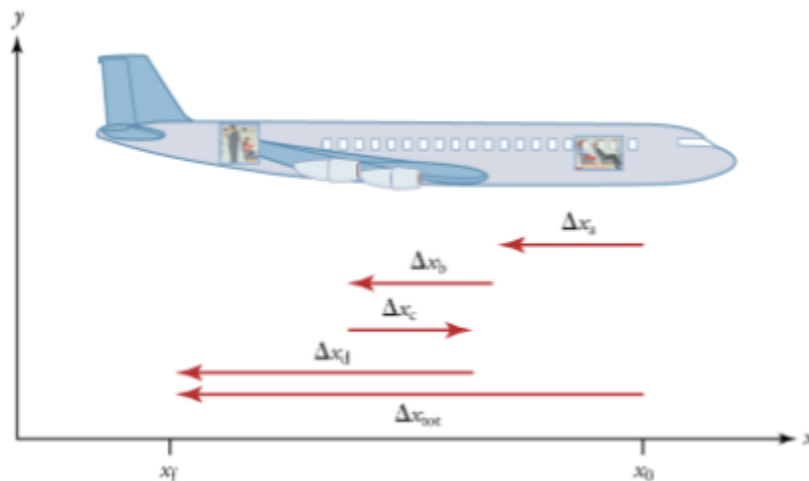


Figure 4: A more detailed record of an airplane passenger heading toward the back of the plane, showing smaller segments of his trip.

The smaller the time intervals considered in a motion, the more detailed the information.

When we carry this process to its logical conclusion, we are left with an infinitesimally small interval. Over such an interval, the average velocity becomes the instantaneous velocity or the velocity at a specific instant. A car's speedometer, for example, shows the magnitude (but not the direction) of the instantaneous velocity of the car. (Police give tickets based on instantaneous velocity, but when calculating how long it will take to get from one place to another on a road trip, you need to use average velocity.) Instantaneous velocity v is the average velocity at a specific instant in time (or over an infinitesimally small-time interval).

Speed:

In everyday language, most people use the terms “speed” and “velocity” interchangeably. In physics, however, they do not have the same meaning and they are distinct concepts. One major difference is that speed has no direction. Thus, speed is a scalar. Just as we need to distinguish between instantaneous velocity and average velocity, we also need to distinguish between instantaneous speed and average speed.

Instantaneous speed is the magnitude of instantaneous velocity. For example, suppose the airplane passenger at one instant had an instantaneous velocity of -3.0 m/s (the minus meaning toward the rear of the plane). At that same time, his instantaneous speed was 3.0 m/s. Or suppose that at one time during a shopping trip your instantaneous velocity is 40 km/h due north. Your instantaneous speed at that instant would be 40 km/h—the same magnitude but without a direction. Average speed, however, is vastly different from average velocity. Average speed is the distance traveled divided by elapsed time.

We have noted that distance traveled can be greater than displacement. So average speed can be greater than average velocity, which is displacement divided by time. For example, if you drive to a store and return home in half an hour, and your car's odometer shows the total distance traveled was 6 km, then your average speed was 12 km/h. Your average velocity, however, was zero, because your displacement for the round trip is zero. (Displacement is change in position and, thus, is zero for a round trip.) Thus, average speed is not simply the magnitude of average velocity.

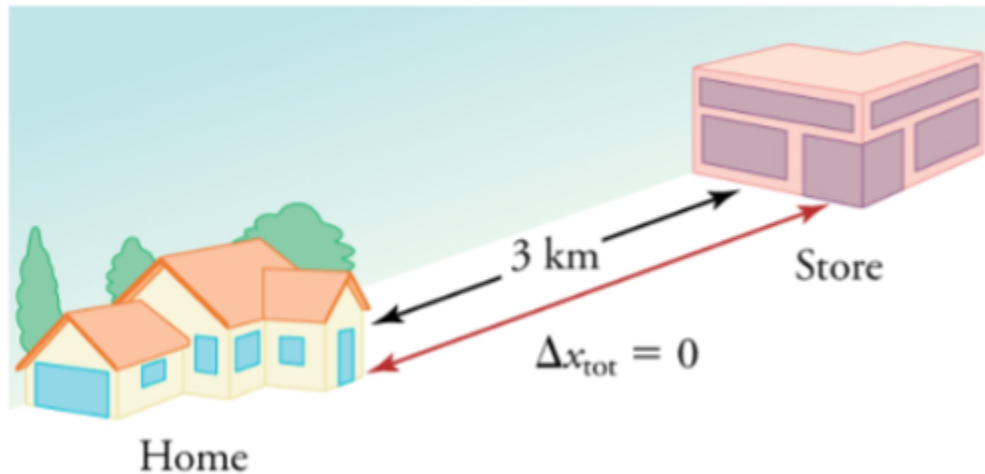


Figure 5: uring a 30-minute round trip to the store, the total distance traveled is 6 km. The average speed is 12 km/h. The displacement for the round trip is zero since there was no net change in position. Thus, the average velocity is zero.

Another way of visualizing the motion of an object is to use a graph. A plot of position or of velocity as a function of time can be extremely useful. For example, for this trip to the store, the position, velocity, and speed-vs.-time graphs are displayed in Figures 6-8. (Note that these graphs depict a very simplified model of the trip. We are assuming that speed is constant during the trip, which is unrealistic given that we will probably stop at the store. But for simplicity's sake, we will model it with no stops or changes in speed. We are also assuming that the route between the store and the house is a perfectly straight line.)

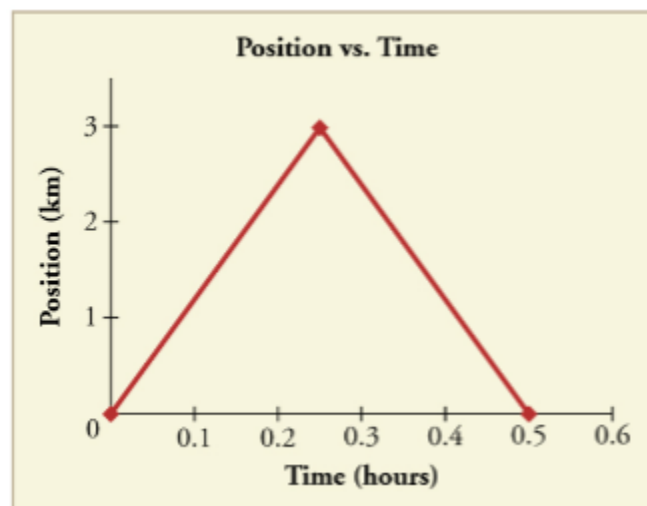


Figure 6: Position vs. time on a trip. Note that the velocity for the return trip is negative.

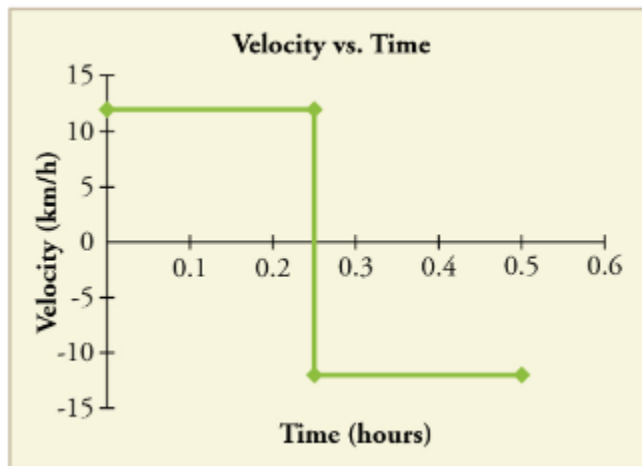


Figure 7: Velocity vs. time on a trip. Note that the velocity for the return trip is negative.

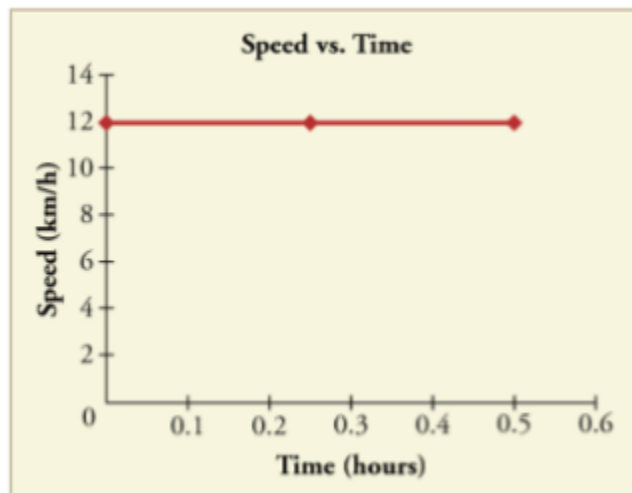


Figure 8: Speed vs. time on a trip. Note that the velocity for the return trip is negative.

Acceleration:

In everyday conversation, to accelerate means to speed up. The accelerator in a car can in fact cause it to speed up. The greater the acceleration, the greater the change in velocity over a given time. The formal definition of acceleration is consistent with these notions, but more inclusive.

Average Acceleration is *the rate at which velocity changes*,

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}$$

Where \bar{a} is average acceleration, v is velocity, and t is time.

Because acceleration is velocity in m/s divided by time in s, the SI units for acceleration are m/s² meters per second squared or meters per second per second, which means by how many meters per second the velocity changes every second.

Recall that velocity is a vector—it has both magnitude and direction. This means that a change in velocity can be a change in magnitude (or speed), but it can also be a change in direction. For example, if a car turns a corner at constant speed, it is accelerating because its direction is changing. The quicker you turn, the greater the acceleration. So, there is an acceleration when velocity changes either in magnitude (an increase or decrease in speed) or in direction, or both.

Keep in mind that although acceleration is in the direction of the change in velocity, it is not always in the direction of motion. When an object's acceleration is in the same direction of its motion, the object will speed up. However, when an object's acceleration is opposite to the direction of its motion, the object will slow down. Speeding up and slowing down should not be confused with a positive and negative acceleration.

Instantaneous Acceleration

Instantaneous acceleration a , or the acceleration at a specific instant in time, is obtained by the same process as discussed for instantaneous velocity, that is by considering an infinitesimally small interval of time. How do we find instantaneous acceleration using only algebra? The answer is that we choose an average acceleration that is representative of the motion. Figure 9 shows graphs of instantaneous acceleration versus time for two quite different motions. In (a) the acceleration varies slightly and the average over the entire interval is nearly the same as the instantaneous acceleration at any time. In this case, we should treat this motion as if it had a constant acceleration equal to the average (in this case about 1.8 m/s^2). In (b) the acceleration varies drastically over time. In such situations it is best to consider smaller time intervals and choose an average acceleration for each. For example, we could consider motion over the time intervals from 0 to 1.0 s and from 1.0 to 3.0 s as separate motions with accelerations of $+3.0 \text{ m/s}^2$ and -2.0 m/s^2 , respectively.

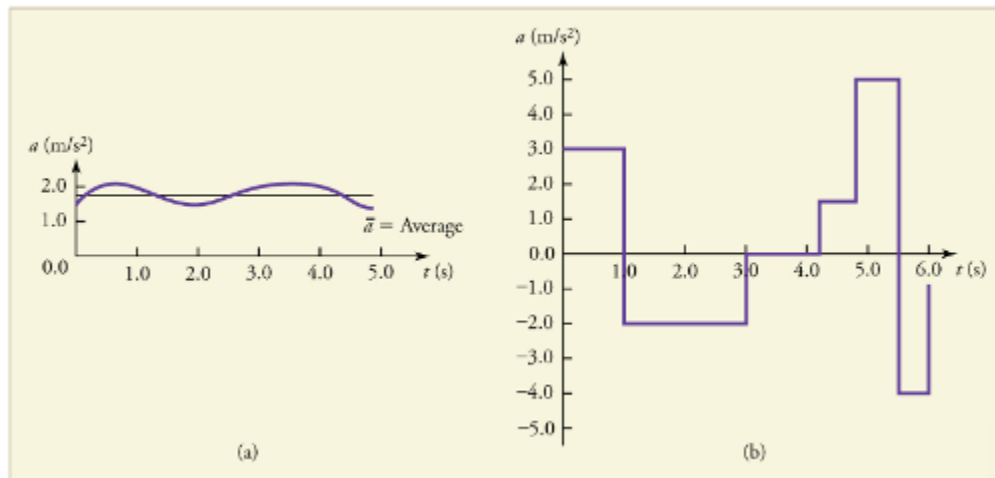


Figure 9: Graphs of instantaneous acceleration versus time for two different one-dimensional motions. (a) Here acceleration varies only slightly and is always in the same direction since it is positive. The average over the interval is nearly the same as the acceleration at any given time. (b) Here the acceleration varies, perhaps representing a package on a post office conveyor belt that is accelerated forward and backward as it bumps along. It is necessary to consider small time intervals (such as from 0 to 1.0 s) with constant or nearly constant acceleration in such a situation.

Check Your Understanding: The Motion of an Object in the Graphic Form through the Time-Space Diagram



An interactive H5P element has been excluded from this version of the text. You can view it online here: <https://uta.pressbooks.pub/oert-mpsfundamentals/?p=544#h5p-119>

Interpreting Motion Graphs

The Moving Man

Please complete the simulation here.

Check Your Understanding: Interpreting Motion Graphs





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Motion Diagram

Comparing Motion Diagrams (Simulation)

Please complete the simulation at this link.

USE KINEMATIC EQUATIONS TO SOLVE FOR DISPLACEMENT, TIME, VELOCITY AND ACCELERATION OF AN OBJECT

In this section, you will learn how to use kinematic equations to solve for displacement, time, velocity and acceleration of an object by reading each description. Also, short problems to check your understanding are included.

MOTION EQUATIONS FOR CONSTANT ACCELERATION IN ONE DIMENSION

We might know that the greater the acceleration of, say, a car moving away from a stop sign, the greater the displacement in a given time. But we have not developed a specific equation that relates acceleration and displacement. In this section, we develop some convenient equations for kinematic relationships.

Notation: t , x , v , a

First, let us make some simplifications in notation. Taking the initial time to be zero, as if time is measured with a stopwatch, is a great simplification. Since elapsed time is $\Delta t = t_f - t_0$, taking $t_0 = 0$ means that $\Delta t = t_f$, the final time on the stopwatch. When initial time is taken to be zero, we use the subscript 0 to denote initial values of position and velocity. That is, x_0 is the initial position and v_0 is the initial velocity. We put no subscripts on the final values. The is, t is the final time, x is the final position, and v is final velocity. This gives a simpler expression for elapsed time – now, $\Delta t = t$. It also simplifies the expression for displacement, which is now $\Delta x = x - x_0$. Also, it simplifies the expression for change in velocity, which is now $\Delta v = v - v_0$. To summarize, using the simplified notation, with the initial time taken to be zero,

$$\Delta t = t$$

$$\Delta x = x - x_0$$

$$\Delta v = v - v_0$$

Where the subscript 0 denotes an initial value and the absence of a subscript denotes a final value in whatever motion is under consideration.

We now make the important assumption that acceleration is constant. This assumption allows us to avoid using calculus to find instantaneous acceleration. Since acceleration is constant, the average and instantaneous accelerations are equal. That is,

$$\bar{a} = a = \text{constant}$$

So, we use the symbol a for acceleration at all times. Assuming acceleration to be constant does not seriously limit the situations we can study nor degrade the accuracy of our treatment. For one thing, acceleration is constant in a considerable number of situations. Furthermore, in many other situations we can accurately describe motion by assuming a constant acceleration equal to the average acceleration for that motion. Finally, in motions where acceleration changes drastically, such as a car accelerating to top speed and then braking to a stop, the motion can be considered in separate parts, each of which has its own constant acceleration.

Solving for Displacement (Δx) and the Final Position (x) from Average Velocity when Acceleration (a) is Constant.

To get our first two new equations, we start with the definition of average velocity:

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

Substituting the simplified notation for Δx and Δt yields

$$\bar{v} = \frac{x - x_0}{t}$$

Solving for x yields

$$x = x_0 + \bar{v}t$$

Where the average velocity is, with constant a

$$\bar{v} = \frac{v_0 + v}{2}$$

The equation $\bar{v} = \frac{v_0 + v}{2}$ reflects the fact that, when acceleration is constant, v is just the simple average of the initial and final velocities. For example, if you steadily increase your velocity (that is, with constant acceleration) from 30 to 60 km/h, then your average velocity during this steady increase is 45 km/h. Using the equation $\bar{v} = \frac{v_0 + v}{2}$, we see that $\bar{v} = \frac{v_0 + v}{2} = \frac{30 \text{ km/h} + 60 \text{ km/h}}{2} = 45 \text{ km/h}$.

The equation $x = x_0 + \bar{v}t$ gives insight into the relationship between displacement, average velocity, and time. It shows, for example, that displacement is a linear function of average velocity. (By linear function, we mean that displacement depends on \bar{v} rather than on \bar{v} raised to some other power, such as \bar{v}^2 . When graphed, linear functions look like straight lines with a constant slope.) On a car trip, for example, we will get twice as far in a given time if we average 90 km/h than if we average 45 km/h.

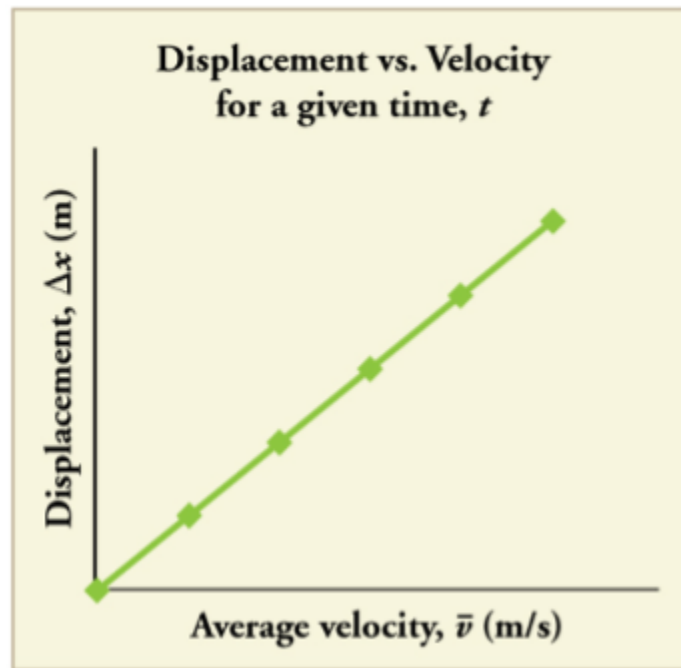


Figure 10: There is a linear relationship between displacement and average velocity. For a given time t , an object moving twice as fast as another object will move twice as far as the other object.

Solving for Final Velocity:

We can derive another useful equation by manipulating the definition of acceleration.

$$a = \frac{\Delta v}{\Delta t}$$

Substituting the simplified notation for Δv and Δt gives us (constant a)

$$a = \frac{v - v_0}{t}$$

Solving for v yields (constant a)

$$v = v_0 + at$$

In addition to being useful in problem solving, the equation $v = v_0 + at$ gives us insight into the relationships among velocity, acceleration, and time. From it we can see, for example that

- Final velocity depends on how large the acceleration is and how long it lasts
- If the acceleration is zero, then the final velocity equals the initial velocity ($v = v_0$), as expected (i.e., velocity is constant)
- If a is negative, then the final velocity is less than the initial velocity.

Solving for Final Position when Velocity is Not Constant ($a \neq 0$)

We can combine the equations above to find a third equation that allows us to calculate the final position of an object experiencing constant acceleration. We start with

$$v = v_0 + at$$

Adding v_0 to each side of this equation and dividing by 2 gives

$$\frac{v_0+v}{2} = v_0 + \frac{1}{2}at$$

Since $\frac{v_0+v}{2} = \bar{v}$ for constant acceleration, then

$$\bar{v} = v_0 + \frac{1}{2}at$$

Now we substitute this expression for \bar{v} into the equation for displacement, $x = x_0 + \bar{v}t$, yielding (constant a)

$$x = x_0 + v_0t + \frac{1}{2}at^2$$

What else can we learn by examining the equation $x = x_0 + v_0t + \frac{1}{2}at^2$? We see that:

- Displacement depends on the square of the elapsed time when acceleration is not zero.
- If acceleration is zero, then the initial velocity equals average velocity ($v_0 = \bar{v}$) and $x = x_0 + v_0t + \frac{1}{2}at^2$ becomes $x = x_0 + v_0t$

Solving for Final Velocity when Velocity is not Constant ($a \neq 0$)

A fourth useful equation can be obtained from another algebraic manipulation of previous equations.

If we solve $v = v_0 + at$ for t , we get

$$t = \frac{v-v_0}{a}$$

Substituting this and $\bar{v} = \frac{v_0+v}{2}$ into $x = x_0 + \bar{v}t$, we get (constant a)

$$v^2 = v_0^2 + 2a(x - x_0)$$

An examination of the equation $v^2 = v_0^2 + 2a(x - x_0)$ can produce further insights into the general relationships among physical quantities:

- The final velocity depends on how large the acceleration is and the distance over which it acts
- For a fixed deceleration, a car that is going twice as fast does not simply stop in twice the distance – it takes much further to stop. (This is why we have reduced speed zones near schools).

Summary of Kinematic Equations (Constant a)

$$x = x_0 + \bar{v}t$$

$$\bar{v} = \frac{v_0 + v}{2}$$

$$v = v_0 + at$$

$$x = x_0 + v_0t + \frac{1}{2}at^2$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

WHAT ARE THE KINEMATIC FORMULAS?

Please read this link on kinematic formulas.

CHOOSING KINEMATIC EQUATIONS

FREE-BODY DIAGRAMS

In this section, you will learn about free-body diagrams and how to make a free-body diagram by reading each description. Also, short problems to check your understanding are included.

INTRODUCTION TO FORCES AND FREE-BODY DIAGRAM

Key terms:

Term	Meaning
Force	A push or pull on an object, usually has symbol F . Has SI units of Newtons (N) or $\frac{kgm}{s^2}$
Contact force	A force that requires contact between objects. Examples are tension, normal force, and friction.
Long range force	A force that does not need contact between objects to exist.
Free body diagram	A diagram showing the forces acting on the object. The object is represented by a dot with forces drawn as arrows pointing away from the dot. Sometimes called force diagrams.

Types of forces:

Force (symbol)	Force type	Description
Weight (F_g or W)	Long range	Force from gravity acting on an object with mass. Sometime called force of gravity. Pulls towards the Earth (down) always.
Tension (F_T or T)	Contact	Force of something pulling on an object. Can be caused by a string, rope, chain, cord, cable, or wire. Pulls along the direction of the rope on the object.
Normal force (F_N or N)	Contact	Force between two objects when they touch. Pushes perpendicularly to the object's surface.
Friction (F_f or f)	Contact	Force resisting sliding between surfaces. Pushes parallel to the contact surface and in the opposite direction of sliding.

How to make a free body diagram:

1. Start by identifying the contact forces. Let us look for what the object is touching by outlining the object (see Figure 11). Draw a dot where something touches the outline; where there is a dot, there must be at least one contact force. Draw the force vectors at the contact points to represent how they push or pull on the object (including correct direction).

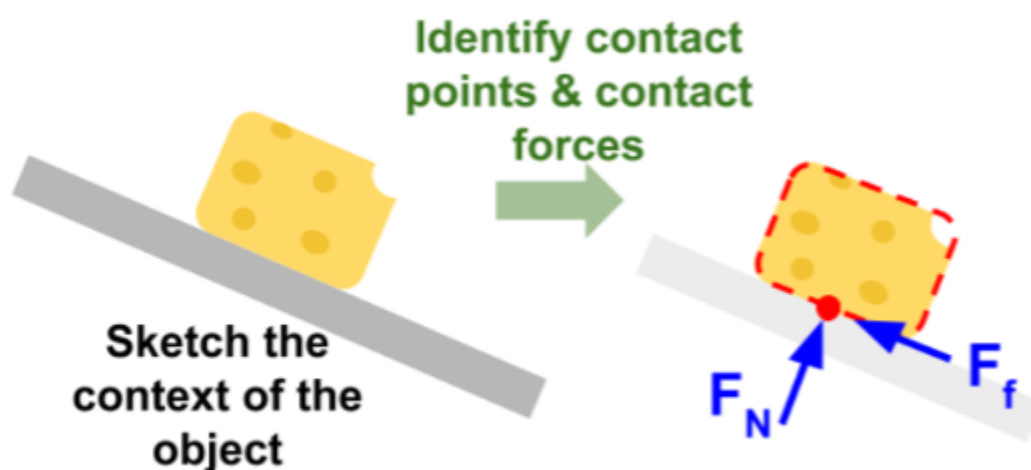


Figure 11: Contact force identification for a block of cheese resting on a ramp.

2. After we have identified the contact forces, draw a dot to represent the object we are interested in (see Figure 12 in example). We only want to find the forces acting on our object and not forces the object exerts on other objects.
3. Draw a coordinate system and label the positive directions. If the object is on an incline, then align the axes with the incline.
4. Draw the contact forces on the dot with an arrow pointing away from the dot. Make sure the arrow lengths are relatively proportional to each other. Label all forces.
5. Draw and label our long-range forces. This will usually be weight unless there is electric charge or magnetism involved.
6. Draw and label your acceleration vector off to the side of the dot – not touching the

dot. If there is no acceleration, then write $a = 0$.

Here is an example of a free body diagram for a block of cheese resting on a table (See Figure 12). Gravity pulls down on the cheese's mass with weight (W) and the table pushes up on the cheese with a normal force (N). Since there are no ropes and the cheese is not trying to slide, there is no tension or friction.

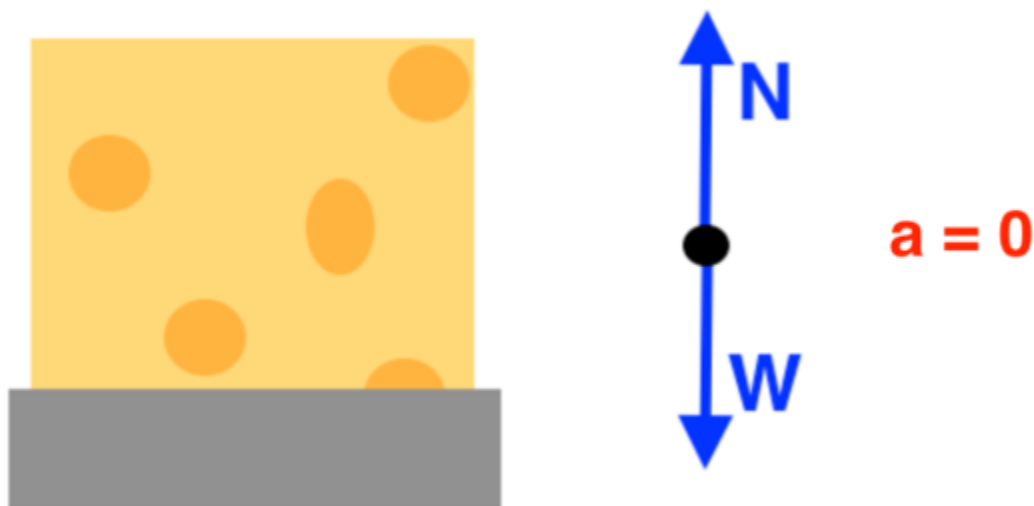


Figure 12: The free body diagram for a block of cheese resting on a table has normal force N up, weight W down, and no acceleration.

Common mistakes and misconceptions

1. Sometimes people draw the forces of the object acting on other things. We only want to draw the forces pushing or pulling on our object. Only focus on what is happening to the object of interest.
2. Sometimes people forget the directions of the different types of forces. Weight is always down, friction is always parallel to the contact surface, normal force is always perpendicular to the contact surface, and tension only pulls.

Check Your Understanding: Forces and Free-body Diagrams



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ESTIMATE THE FRICTION FORCE AND NORMAL REACTION ON A STATIC OBJECT

In this section, you will learn how estimate the friction force and normal reaction on a static object and learn about kinetic and static friction forces by reading each description along with watching the videos. Also, short problems to check your understanding are included.

NORMAL FORCE

If the force supporting a load is perpendicular to the surface of contact between the load and its support, this force is defined to be a normal force and is given the symbol N (different from newton). The word normal means perpendicular to a surface.

FRICTION

Friction is a force that opposes relative motion between systems in contact. One of the simpler characteristics of friction is that it is parallel to the contact surface between systems and always in a direction that opposes motion or attempted motion of the systems relative to each other. If two systems are in contact and moving relative to one another, then the friction between them is called kinetic friction. For example, friction slows a hockey puck sliding on ice. But when objects are stationary, static friction can act between them; the static friction is usually greater than the kinetic friction between the objects.

As seen in Table 1 below, the coefficients of kinetic friction are less than their static counterparts. The equations explained below include the dependence of friction on materials and the normal force.

Table 1: Coefficients of Static and Kinetic Friction

System	μ_s Static Function	μ_k Kinetic Function
Rubber on dry concrete	1.0	0.7
Rubber on wet concrete	0.7	0.5
Wood on wood	0.5	0.3
Waxed wood on wet snow	0.14	0.1
Metal on wood	0.5	0.3
Steel on steel (dry)	0.6	0.3
Steel on steel (oiled)	0.05	0.03
Teflon on steel	0.04	0.04
Bone lubricated by synovial fluid	0.016	0.015
Shoes on wood	0.9	0.7
Shoes on ice	0.1	0.05
Ice on ice	0.1	0.03
Steel on ice	0.04	0.02

Kinetic and Static Friction Forces



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Static Friction Equation



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Where there is no motion between the objects, the magnitude of static friction is $f_s \leq \mu_s N$, where μ_s is the coefficient of static friction and N is the magnitude of the normal force.

The symbol \leq means less than or equal to, implying that static friction can have a minimum and a maximum value of $\mu_s N$. Static friction is a responsive force that increases to be equal and opposite to whatever force is exerted, up to its maximum limit. Once the applied force exceeds $f_{s(\max)}$, the object will move. Thus $f_{s(\max)} = \mu_s N$.

Kinetic Friction Equation



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Once an object is moving, the magnitude of kinetic friction f_k is given by $f_k = \mu_k N$, where μ_k is the coefficient of kinetic friction.

Check Your Understanding: Friction



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STATIC AND KINETIC FRICTION EXAMPLE



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Check Your Understanding: Static and Kinetic Friction



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FORCES AND NEWTON'S LAWS

In this section, you will learn about Newton's three laws of motion along with laws of gravitation by reading each description along with watching the videos. Also, short problems to check your understanding are included.

NEWTON'S 3 LAWS OF MOTION



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Newton's First Law of Motion

There exists an inertial frame of reference such that a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force.

The first law of motion postulates the existence of at least one frame of reference which we call an inertial reference frame, relative to which the motion of an object not subject to forces is a straight line at a constant speed. An inertial reference frame is any reference frame that is not itself accelerating. A car traveling at constant velocity is an inertial reference frame. A car slowing down for a stoplight, or speeding up after the light turns green, will be accelerating and is not an inertial reference frame. Finally, when the car goes around a turn, which is due to an acceleration changing the direction of the velocity vector, it is not an inertial reference frame. Note that Newton's laws of motion are only valid for inertial reference frames.

Rather than contradicting our experience, Newton's first law of motion states that there must be a cause (which is a net external force) for there to be any change in velocity (either a change in magnitude or direction) in an inertial reference frame. An object sliding across a table or floor slows down due to the net force of friction acting on the object.

The idea of cause and effect is crucial in accurately describing what happens in various situations. For example, consider what happens to an object sliding along a rough horizontal surface. The object quickly grinds to a halt. If we spray the surface with talcum powder to make the surface smoother, the object slides farther. If we make the surface even smoother by rubbing lubricating oil on it, the object slides farther yet. Extrapolating to a frictionless surface, we can imagine the object sliding in a straight line indefinitely. Friction is thus the cause of the slowing (consistent with Newton's first law). The object would not slow down at all if friction were completely eliminated. Consider an air hockey table. When the air is turned off, the puck slides only a short distance before friction slows it to a stop. However, when the air is turned on, it creates a nearly frictionless surface, and the puck glides long distances without slowing down. Additionally, if we

know enough about the friction, we can accurately predict how quickly the object will slow down. Friction is an external force.

Mass: The property of a body to remain at rest or to remain in motion with constant velocity is called inertia. Newton's first law is often called the law of inertia. As we know from experience, some objects have more inertia than others. It is obviously more difficult to change the motion of a large boulder than that of a basketball, for example. The inertia of an object is measured by its mass.

An object with a small mass will exhibit less inertia and be more affected by other objects. An object with a large mass will exhibit greater inertia and be less affected by other objects. This inertial mass of an object is a measure of how difficult it is to alter the uniform motion of the object by an external force.

Roughly speaking, mass is a measure of the amount of “stuff” (or matter) in something. The quantity or amount of matter in an object is determined by the numbers of atoms and molecules of several types it contains. Unlike weight, mass does not vary with location. The mass of an object is the same on Earth, in orbit, or on the surface of the Moon. In practice, it is exceedingly difficult to count and identify all of the atoms and molecules in an object, so masses are not often determined in this manner. Operationally, the masses of objects are determined by comparison with the standard kilogram.

Check Your Understanding: Newton's First Law of Motion



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Newton's Second Law of Motion

Newton's second law of motion is closely related to Newton's first law of motion. It mathematically states the cause-and-effect relationship between force and changes in motion. Newton's second law of motion is more quantitative and is used extensively to calculate what happens in situations involving a force. Before we can write down Newton's second law as a simple equation giving the exact relationship of force, mass, and acceleration, we need to sharpen some ideas that have already been mentioned.

First, what do we mean by a change in motion? The answer is that a change in motion is equivalent to a change in velocity. A change in velocity means, by definition, that there is an acceleration. Newton's first law says that a net external force causes a change in motion; thus, we see that a net external force causes acceleration.

Another question immediately arises. What do we mean by an external force? An intuitive notion of external is correct—an external force acts from outside the system of interest. An internal force acts between elements of the system. Only external forces affect the motion of a system, according to Newton’s first law.

To obtain an equation for Newton’s Second Law, we first write the relationship of acceleration and net external force as the proportionality

$$a \propto F_{\text{net}}$$

Where the symbol \propto means “proportional to,” and F_{net} is the net external force. This proportionality states acceleration is directly proportional to the net external force. Now, it also seems reasonable that acceleration should be inversely proportional to the mass of the system. In other words, the larger the mass (the inertia), the smaller the acceleration produced by a given force. This proportionality is written as $a \propto \frac{1}{m}$, where m is the mass of the system. Combining the two proportionalities just given yield’s Newton’s Second Law of Motion.

Newton’s Second Law of Motion: The acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system, and inversely proportional to its mass. In equation form, Newton’s Second Law of Motion is:

$$a = \frac{F_{\text{net}}}{m}$$

Often written as:

$$F_{\text{net}} = ma$$

Units of Force: $F_{\text{net}} = ma$ is used to define the units of force in terms of the three basic units for mass, length, and time. The SI unit of force is called the newton (abbreviated N) and is the force needed to accelerate a 1-kg system at the rate of 1 m/s^2 . That is, since $F_{\text{net}} = ma$, $1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$.

Weight and the Gravitational Force: When an object is dropped, it accelerates toward the center of Earth. Newton’s second law states that a net force on an object is responsible for its acceleration. If air resistance is negligible, the net force on a falling object is the gravitational force, commonly called its **weight w**.

Since the object experiences only the downward force of gravity, $F_{\text{net}} = w$. We know that the acceleration of an object due to gravity is g , or $a = g$. Substituting these into Newton’s Second Law gives $w = mg$. Since $g = 9.80 \text{ m/s}^2$ on Earth, the weight of a 1.0 kg object on Earth is 9.8N, as we see: $w = mg = (1.0\text{kg}) (9.80\frac{\text{m}}{\text{s}^2}) = 9.8\text{N}$.

When the net external force on an object is its weight, we say that it is in free-fall. That is, the only force acting on the object is the force of gravity. In the real world, when objects fall downward toward Earth, they are never truly in free-fall because there is always some upward force from the air acting on the object.

Common misconceptions: Mass vs. Weight: It is important to be aware that weight and mass are very different physical quantities, although they are closely related. Mass is the quantity of matter (how much “stuff”) and does not vary in classical physics, whereas weight is the gravitational force and does vary depending on gravity. It is tempting to equate the two, since most of our examples take place on Earth, where the weight of an object only varies a little with the location of the object.

Check Your Understanding: Newton’s Second Law of Motion



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Newton’s Third Law of Motion

Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that it exerts.

This law represents a certain symmetry in nature: Forces always occur in pairs, and one body cannot exert a force on another without experiencing a force itself. We sometimes refer to this law loosely as “action-reaction,” where the force exerted is the action and the force experienced as a consequence is the reaction. Newton’s third law has practical uses in analyzing the origin of forces and understanding which forces are external to a system. You might think that two equal and opposite forces would cancel, but they do not because they act on different systems.

Other examples of Newton’s third law are easy to find. As a professor paces in front of a whiteboard, she exerts a force backward on the floor. The floor exerts a reaction force forward on the professor that causes her to accelerate forward. Similarly, a car accelerates because the ground pushes forward on the drive wheels in reaction to the drive wheels pushing backward on the ground. You can see evidence of the wheels pushing backward when tires spin on a gravel road and throw rocks backward.

In another example, rockets move forward by expelling gas backward at high velocity. This means the rocket exerts a large backward force on the gas in the rocket combustion chamber, and the gas therefore exerts a large reaction force forward on the rocket. This reaction force is called thrust. It is a common misconception that rockets propel themselves by pushing on the ground or on the air behind them. They actually work better in a vacuum, where they can more readily expel the exhaust gases. Helicopters similarly create lift by pushing air down, thereby experiencing an upward reaction force.

Birds and airplanes also fly by exerting force on air in a direction opposite to that of whatever force they need.

Check Your Understanding: Newton's Third Law of Motion



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GRAVITY FORCE LAB



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LAWS OF GRAVITATION



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Newton's Universal Law of Gravitation

The gravitational force is relatively simple. It is always attractive, and it depends only on the masses involved and the distance between them. Stated in modern language, **Newton's universal law of gravitation** states that every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

Misconception alert: The magnitude of the force on each object (one has larger mass than the other) is the same, consistent with Newton's Third Law.

For two bodies having masses m and M with a distance r between their centers of mass, the equation for Newton's Universal Law of Gravitation is

$$F = G \frac{mM}{r^2}$$

Where F is the magnitude of the gravitation force and G is a proportionality factor called the gravitational constant. G is a universal gravitational constant—that is, it is thought to be the same everywhere in the universe. It has been measured experimentally to be in SI units,

$$G = 6.673 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$$

INTRODUCTION TO SPATIAL INTERACTION MODELLING



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Check Your Understanding: Spatial Interaction Modelling



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DESCRIBE THE MOTION OF AN OBJECT ALONG A CIRCULAR PATH

In this section, you will learn about the motion of an object along a circular path by reading each description along with watching the videos. Also, short problems to check your understanding are included.

ANGULAR ROTATION

Angle Units





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Angular Position and Displacement



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Angular Velocity and Acceleration



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Uniform Circular Motion



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Rotation Angle and Angular Velocity

We begin the study of uniform circular motion by defining two angular quantities needed to describe rotational motion.

Rotation Angle

When objects rotate about some axis – for example, when the CD in Figure 13 below rotates about its center – each point in the object follows a circular arc. Consider a line from the center of the CD to its edge. Each pit used to record sound along this line moves through the same angle in the same amount of time. The rotation angle is the amount of rotation and is analogous to linear distance. We define the **rotation angle** $\Delta\theta$ to be the ratio of the arc length to the radius of curvature:

$$\Delta\theta = \frac{\Delta s}{r}$$



Figure 13: All points on a CD travel in circular arcs. The pits along a line from the center to the edge all move through the same angle $\Delta\theta$ in a time Δt

The **arc length** ΔS is the distance traveled along a circular path as shown in Figure 14 below. Note that r is the **radius of curvature** of the circular path.

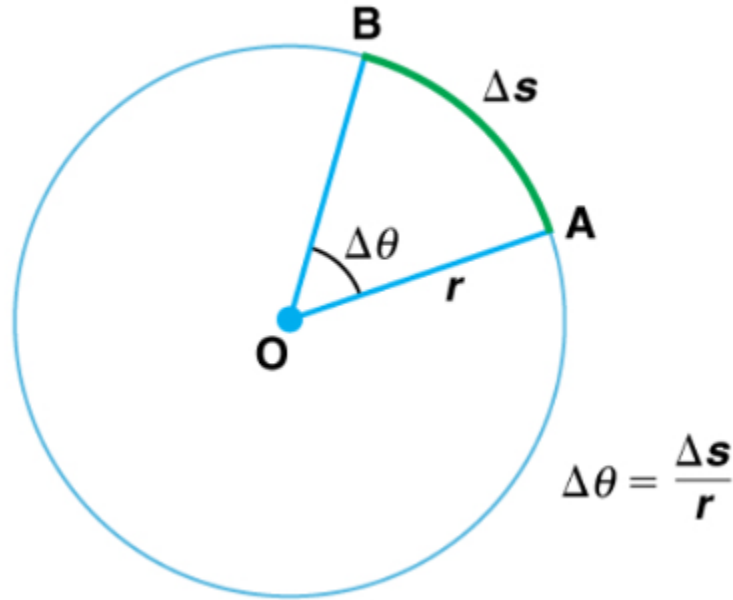


Figure 14: The radius of a circle is rotated through an angle $\Delta\theta$. The arc length Δs is described on the circumference.

We know that for one complete revolution, the arc length is the circumference of a circle of radius r . The circumference of a circle is $2\pi r$. Thus, for one complete revolution the rotation angle is

$$\Delta\theta = \frac{2\pi r}{r} = 2\pi$$

This result is the basis for defining the units used to measure rotation angles, $\Delta\theta$ to be **radians (rad)**, defined so that

$$2\pi \text{ rad} = 1 \text{ revolution}$$

A comparison of some useful angles expressed in both degrees and radians is shown in Table 2 below.

Table 2: Comparison of Angular Units

Degree Measures	Radian Measure
30°	$\frac{\pi}{6}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
120°	$\frac{2\pi}{3}$
135°	$\frac{3\pi}{4}$
180°	π

Angular Velocity

How fast is an object rotating? We define **angular velocity** ω as the rate of change of an angle. In symbols, this is

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Where an angular rotation $\Delta\theta$ takes place in a time Δt . The greater the rotation angle in a given amount of time, the greater the angular velocity. The units for angular velocity are radians per second (rad/s).

Angular velocity ω is analogous to linear velocity v . To get the precise relationship between angular and linear velocity, we again consider a pit on the rotating CD. This pit moves an arc length Δs in a time Δt , and so it has a linear velocity

$$v = \frac{\Delta s}{\Delta t}$$

From $\Delta\theta = \frac{\Delta s}{r}$ we see that $\Delta s = r\Delta\theta$. Substituting this into the expression for v gives

$$v = \frac{r\Delta\theta}{\Delta t} = r\omega$$

We write this relationship in two different ways and gain two different insights:

$$v = r\omega \text{ or } \omega = \frac{v}{r}$$

The first relationship in $v = r\omega$ or $\omega = \frac{v}{r}$ states that the linear velocity v is proportional to the distance from the center of rotation, thus, it is the largest for a point on the rim (largest r), as you might expect. We can also call this linear speed v of a point on the rim the *tangential speed*. The second relationship in $v = r\omega$ or $\omega = \frac{v}{r}$ can be illustrated by considering the tire of a moving car. Note that the speed of a point on the rim of the tire is the same as the speed v of the car. See Figure 15 below. So, the faster the car moves, the faster the tire spins – large v means a large ω , because $v = r\omega$. Similarly, a larger-radius tire rotating at the same angular velocity (ω) will produce a greater linear speed (v) for the car.

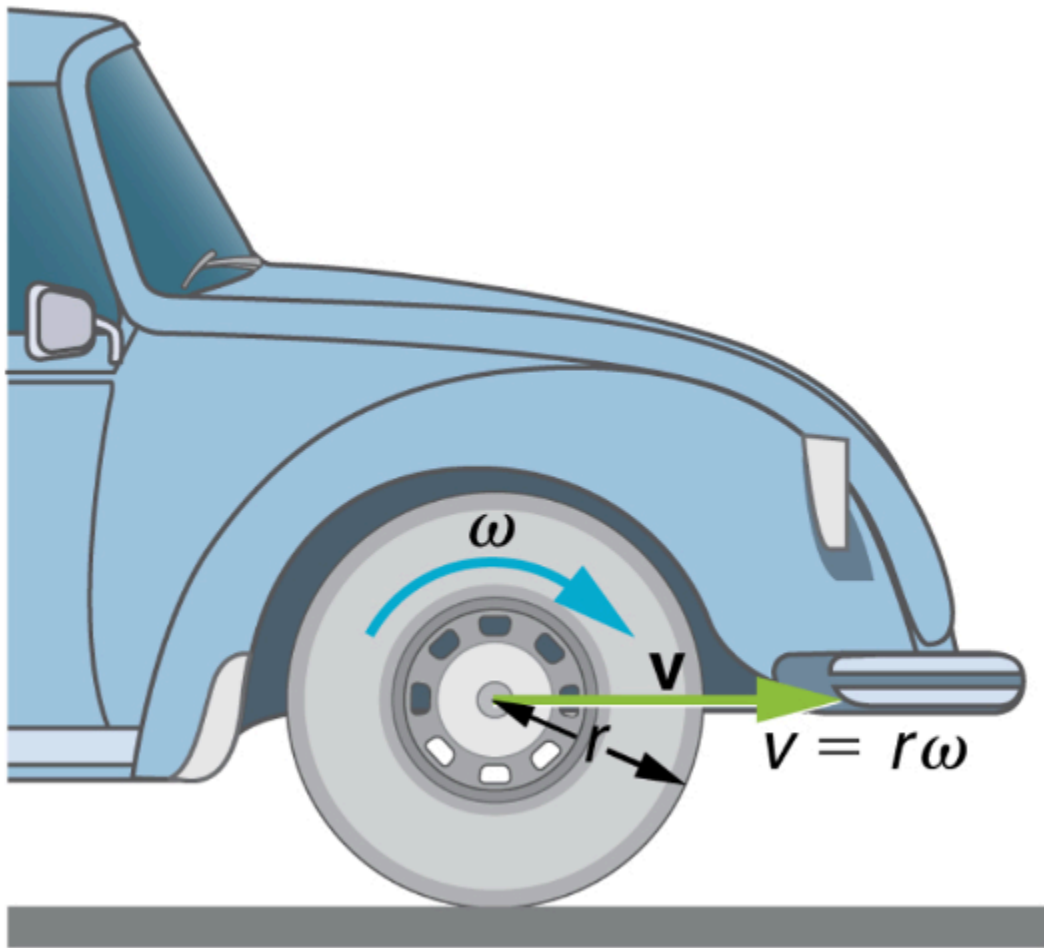


Figure 15: A car moving at a velocity v to the right has a tire rotating with an angular velocity ω . The speed of the tread of the tire relative to the axle is v , the same as if the car were jacked up. Thus, the car moves forward at linear velocity $v = r\omega$, where r is the tire radius. A larger angular velocity for the tire means a greater velocity for the car.

Check Your Understanding: Angular Rotation



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CIRCULAR MOTION AND CENTRIPETAL ACCELERATION

Centripetal Force and Acceleration Intuition



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Visual Understanding of Centripetal Acceleration Formula



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Centripetal Acceleration

We know from kinematics that acceleration is a change in velocity, either in its magnitude or in its direction, or both. In uniform circular motion, the direction of the velocity changes constantly, so there is always an associated acceleration, even though the magnitude of the velocity might be constant. You experience this acceleration yourself when you turn a corner in your car. (If you hold the wheel steady during a turn and move at constant speed, you are in uniform circular motion.) What you notice is a sideways acceleration because you and the car are changing direction. The sharper the curve and the greater your speed, the more noticeable this acceleration will become. In this section we examine the direction and magnitude of that acceleration.

Figure 16 below shows an object moving in a circular path at constant speed. The direction of the instantaneous velocity is shown at two points along the path. Acceleration is in the direction of the change in velocity, which points directly toward the center of rotation (the center of the circular path). This pointing is shown with the vector diagram in the figure. We call the acceleration of an object moving in uniform circular motion (resulting from a net external force) the centripetal acceleration (a_c); centripetal means “toward the center” or “center seeking.”

$$\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$$

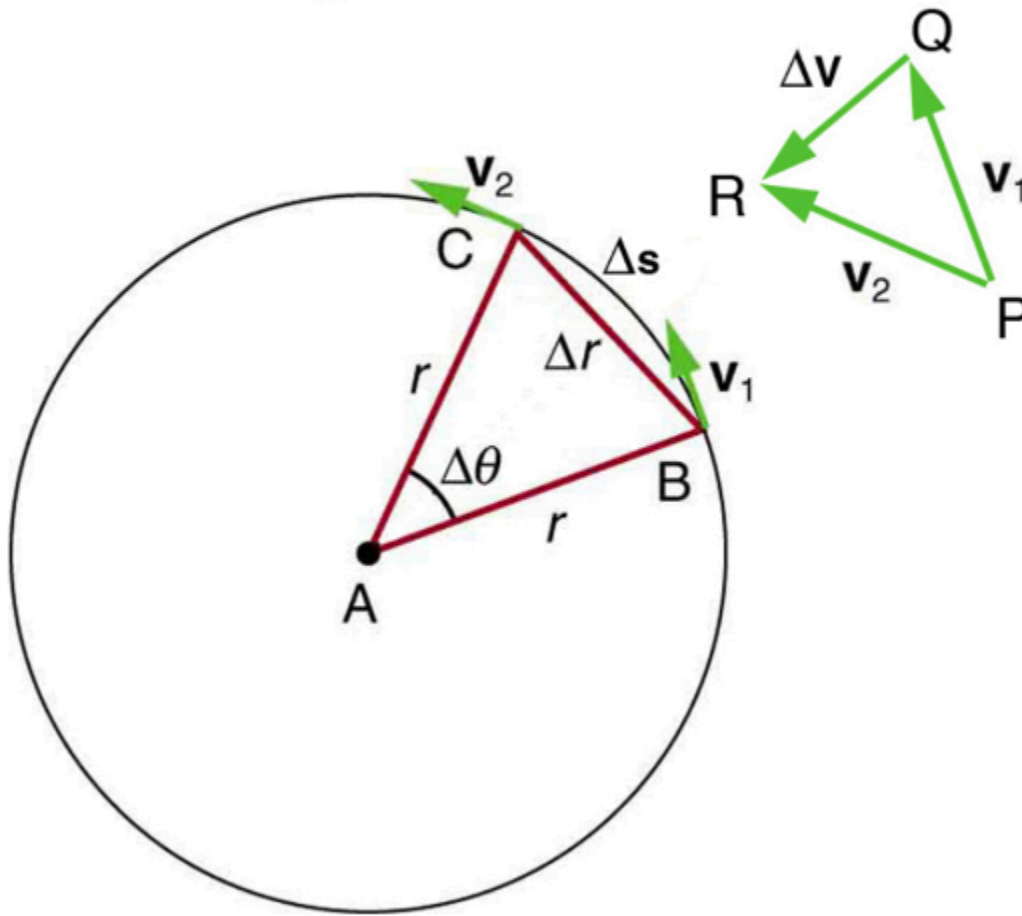


Figure 16: The directions of the velocity of an object at two different points are shown, and the change in velocity Δv is seen to point directly toward the center of curvature. (See small inset). Because $a_c = \frac{\Delta v}{\Delta t}$, the acceleration is also toward the center; a_c is called centripetal acceleration. (Because $\Delta\theta$ is exceedingly small, the arc length Δs is equal to the chord length Δr for small time differences).

The direction of centripetal acceleration is toward the center of curvature, but what is its magnitude? Note that the triangle formed by the velocity vectors and the one formed by the radii r and Δs are similar. Both the triangles ABC and PQR are isosceles triangles (two equal sides). The two equal sides of the velocity vector triangle are the speeds $v_1 = v_2 = v$. Using the properties of two similar triangles, we obtain

$$\frac{\Delta v}{v} = \frac{\Delta s}{r}$$

Acceleration is $\frac{\Delta v}{\Delta t}$, and so we first solve this expression for Δv :

$$\Delta v = \frac{v}{r} \Delta s$$

Then we divide this by Δt , yielding

$$\frac{\Delta v}{\Delta t} = \frac{v}{r} \times \frac{\Delta s}{\Delta t}$$

Finally, noting that $\frac{\Delta v}{\Delta t} = a_c$, and that $\frac{\Delta s}{\Delta t} = v$, the linear or tangential speed, we see that the magnitude of the centripetal acceleration is

$$a_c = \frac{v^2}{r}$$

Which is the acceleration of an object in a circle of radius r at a speed v . So, centripetal acceleration is greater at high speeds and in sharp curves (smaller radius), as you have noticed when driving a car. But it is a bit surprising that a_c is proportional to speed squared, implying, for example, that it is four times as hard to take a curve at 100 km/h than at 50 km/h. A sharp corner has a small radius, so that a_c is greater for tighter turns, as you have probably noticed.

It is also useful to express a_c in terms of angular velocity. Substituting $v = r\omega$ into the above expression, we find $a_c = \frac{(r\omega)^2}{r} = r\omega^2$. We can express the magnitude of centripetal acceleration using either of the two equations:

$$a_c = \frac{v^2}{r}; a_c = r\omega^2$$

Recall that the direction of a_c is toward the center. You may use whichever expression is more convenient.

Check Your Understanding: Circular Motion and Centripetal Acceleration



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CENTRIPETAL FORCES

Centripetal Force Problem Solving



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Centripetal Force

Any force or combination of forces can cause a centripetal or radial acceleration. Just a

few examples are the tension in the rope on a tether ball, the force of Earth's gravity on the Moon, friction between roller skates and a rink floor, a banked roadway's force on a car, and forces on the tube of a spinning centrifuge.

Any net force causing circular motion is called a **centripetal force**. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration. According to Newton's Second Law of Motion, the net force is mass times acceleration: net $F = ma$. For uniform circular motion, the acceleration is the centripetal acceleration – $a = a_c$. Thus, the magnitude of centripetal force F_c is

$$F_c = ma_c$$

By using the expressions for centripetal acceleration a_c from $a_c = \frac{v^2}{r}$; $a_c = r\omega^2$, we get two expressions for the centripetal force F_c in terms of mass, velocity, angular velocity, and radius of curvature:

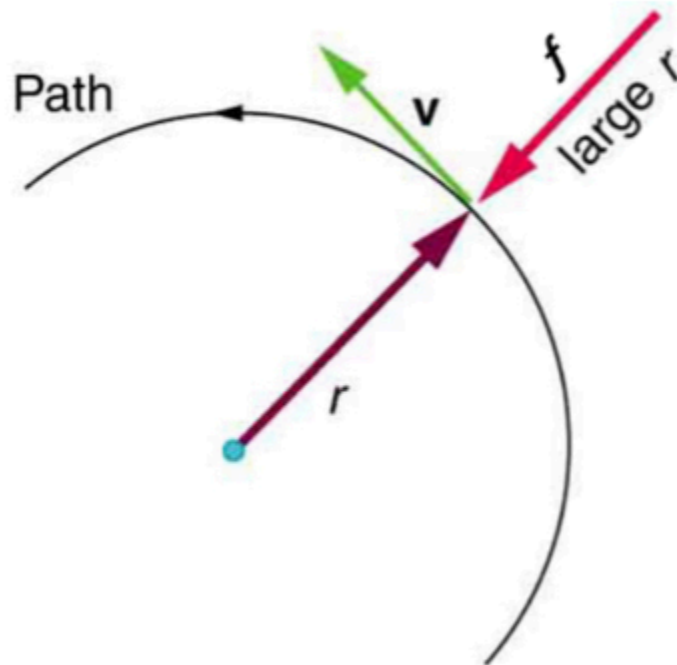
$$F_c = m\frac{v^2}{r}; F_c = mr\omega^2$$

You may use whichever expression for centripetal force is more convenient. Centripetal force F_c is always perpendicular to the path and pointing to the center of curvature, because a_c is perpendicular to the velocity and pointing to the center of curvature.

Note that if you solve the first expression for r , you get

$$r = \frac{mv^2}{F_c}$$

This implies that for a given mass and velocity, a large centripetal force causes a small radius of curvature – that is, a tight curve.



$f = F_c$ is parallel to a_c since $F_c = ma_c$

Figure 17: The frictional force supplies the centripetal force and is numerically equal to it. Centripetal force is perpendicular to velocity and causes uniform circular motion. The larger the F_c , the smaller the radius of curvature r and the sharper the curve.

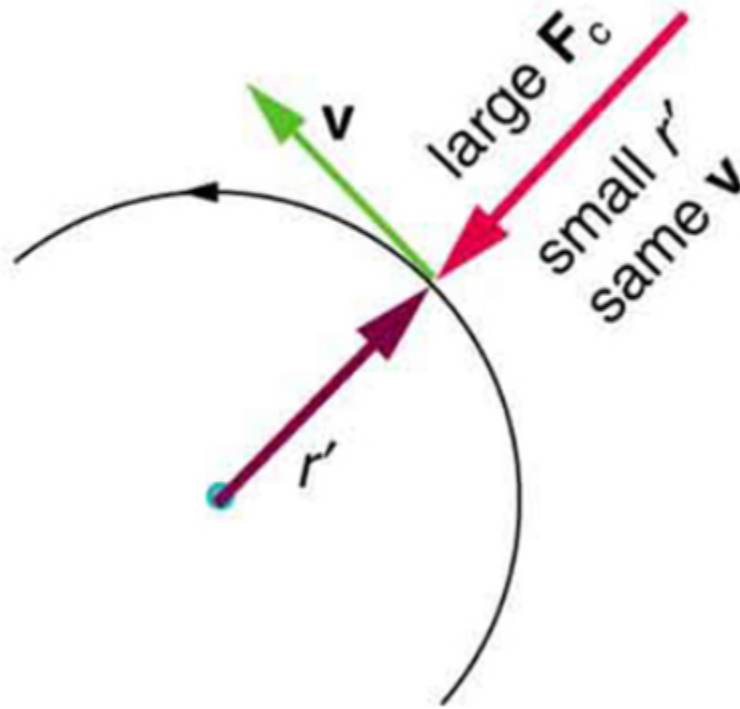


Figure 18: The frictional force supplies the centripetal force and is numerically equal to it. Centripetal force is perpendicular to velocity and causes uniform circular motion. The larger the F_c the smaller the radius of curvature r and the sharper the curve. This curve has the same v , but a larger F_c produces a smaller r .

Example 1

1. What coefficient of friction do car tires need on a flat curve?
 - (a) Calculate the centripetal force exerted on a 900 kg car that negotiates a 500 m radius curve as 25.0 m/s.
 - (b) Assuming an unbanked curve, find the minimum static coefficient of friction, between the tires and the road, static friction being the reason that keeps the car from slipping (See Figure 19 below).

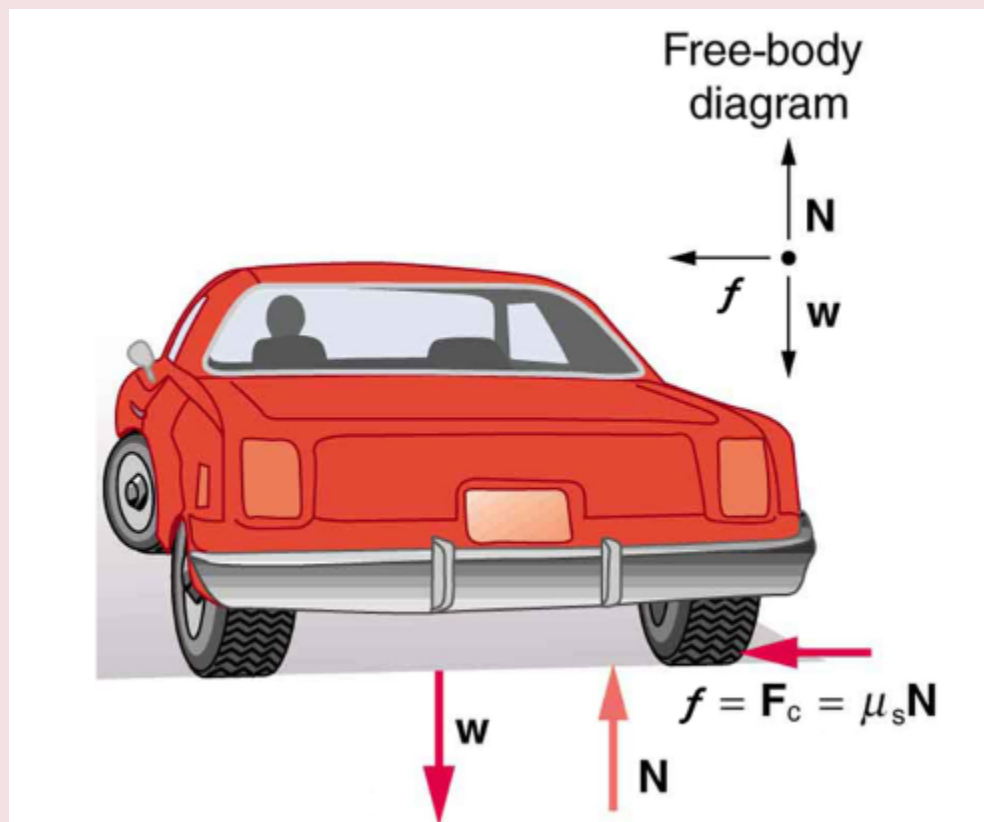


Figure 19

Answer: (a) $F_c = \frac{mv^2}{r} = \frac{(900 \text{ kg})(25.0 \text{ m/s})^2}{(500 \text{ m})} = 1125 \text{ N}$

(b) Figure 19 shows the forces acting on the car on an unbanked (level ground) curve. Friction is to the left, keeping the car from slipping, and because it is the only horizontal force acting on the car, the friction is the centripetal force in this case. We know that the maximum static friction (at which the tires roll but do not slip) is $\mu_s N$, where μ_s is the static coefficient of friction and N is the normal force. The normal force equals the car's weight on level ground, so that $N = mg$. Thus the centripetal force in this situation is

$$F_c = f = \mu_s N = \mu_s mg$$

Now we have a relationship between centripetal force and the coefficient of friction. Using the first expression for F_c from the equation

$$\left. \begin{array}{l} F_c = m \frac{v^2}{r} \\ F_c = mr\omega^2 \end{array} \right\}$$

$$m \frac{v^2}{r} = \mu_s mg$$

We solve this for μ_s , noting that mass cancels, and obtain

$$\mu_s = \frac{v^2}{rg}$$

Substituting the knowns,

$$\mu_s = \frac{(25.0 \text{ m/s})^2}{(500 \text{ m})(9.80 \text{ m/s}^2)} = 0.13$$

Let us now consider **banked curves**, where the slope of the road helps you negotiate the curve. See Figure 20 below. The greater the angle θ , the faster you can take the curve. Racetracks for bikes as well as cars, for example, often have steeply banked curves. In an “ideally banked curve,” the angle θ is such that you can negotiate the curve at a certain speed without the aid of friction between the tires and the road. We will derive an expression for θ for an ideally banked curve and consider an example related to it.

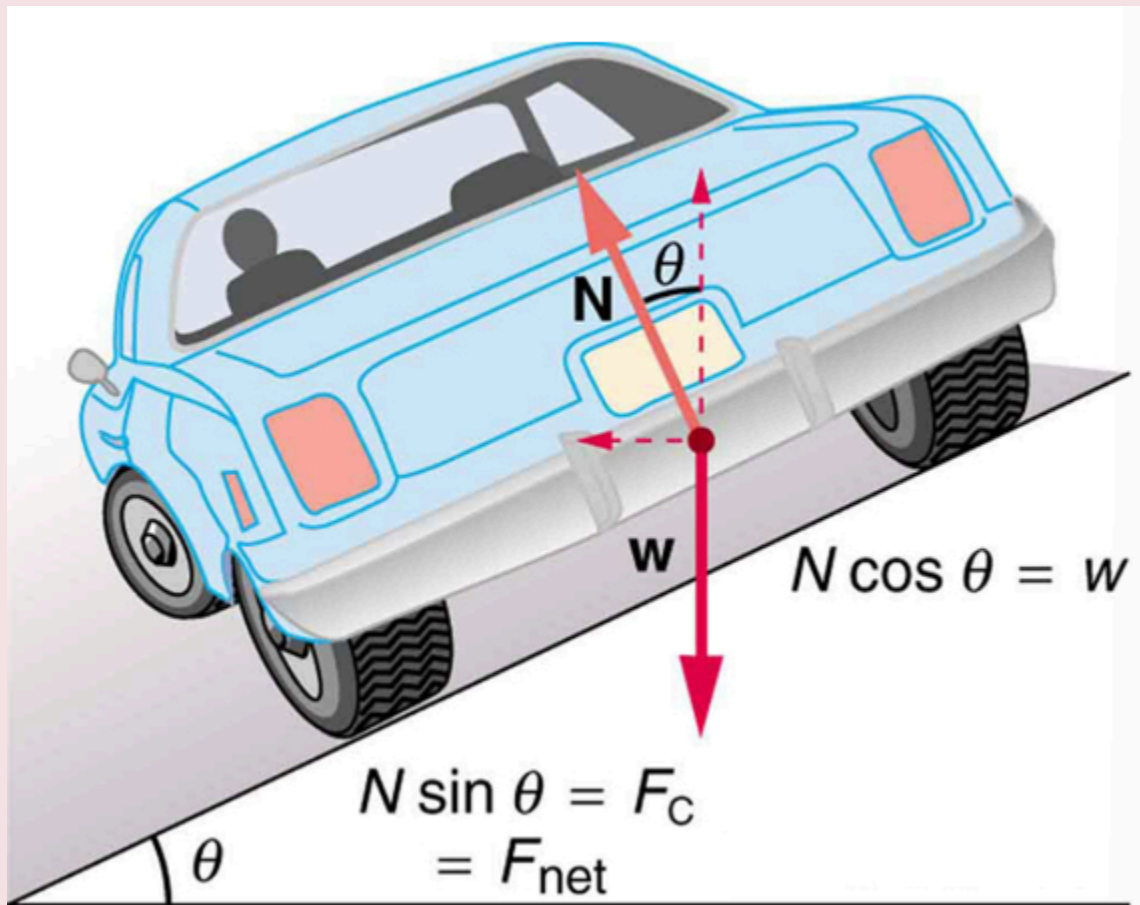


Figure 20: The car on this banked curve is moving away and turning to the left.

For **ideal banking**, the net external force equals the horizontal centripetal force in the absence of friction. The components of the normal force N in the horizontal and vertical directions must equal the centripetal force and the weight of the car, respectively. In cases in which forces are not parallel, it is most convenient to consider components along perpendicular axes – in this case, the vertical and horizontal directions.

Figure 20 above shows a free body diagram for a car on a frictionless banked curve. If the angle θ is ideal for the speed and radius, then the net external force will equal the necessary centripetal force. The only two external forces acting on the car are its weight w and the normal force of the road N . (A frictionless surface can only exert a force perpendicular to the surface – that is, a normal force). These two forces must add to give a net external force that is horizontal toward the center of curvature and has magnitude mv^2/r . Because this is the crucial force and it is horizontal, we use a coordinate system with vertical and horizontal axes. Only the normal force has a horizontal component, and so this must equal the centripetal force – that is,

$$N \sin \theta = \frac{mv^2}{r}$$

Because the car does not leave the surface of the road, the net vertical force must be zero, meaning that the vertical components of the two external forces must be equal in magnitude and opposite in direction. From the figure, we see that the vertical component of the normal force is $N \cos \theta$, and the only other vertical force is the car's weight. These must be equal in magnitude; thus,

$$N \cos \theta = mg$$

Now we can combine the last two equations to eliminate N and get an expression for θ , as desired. Solving the second equation for $N = mg/(\cos \theta)$, and substituting this into the first yields

$$mg \frac{\sin \theta}{\cos \theta} = \frac{mv^2}{r}$$

$$mg \tan(\theta) = \frac{mv^2}{r}$$

$$\tan \theta = \frac{v^2}{rg}$$

Taking the inverse tangent gives

$$\theta = \tan^{-1} \left(\frac{v^2}{rg} \right); \text{ (ideally banked curve, no friction)}$$

This expression can be understood by considering how θ depends on v and r . A large θ will be obtained for a large v and a small r . That is, roads must be steeply banked for high speeds and sharp curves. Friction helps, because it allows you to take the curve at greater or lower speed than if the curve is frictionless. Note that θ does not depend on the mass of the vehicle

Example 2

1. What is the ideal speed to take a steeply banked tight curve? Curves on some test tracks and racecourses, such as the Daytona International Speedway in Florida, are very steeply banked. This banking, with the aid of tire friction and very stable car configurations, allows the curves to be taken at very high speed. To illustrate, calculate the speed at which a 100 m radius curve banked at 65.0° should be driven if the road is frictionless.

Answer: starting with $\tan \theta = \frac{v^2}{rg}$, we get $v = (rg \tan \theta)^{\frac{1}{2}}$

Noting that $\tan 65.0^\circ = 2.14$, we obtain

$$v = \left[(100\text{m}) \left(\frac{9.80\text{m}}{\text{s}^2} \right) (2.14) \right]^{\frac{1}{2}} = 45.8 \text{ m/s}$$

FRAMES OF REFERENCE

Fictitious Forces



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Fictitious Forces and Non-Inertial Frames: The Coriolis Force

What do taking off in a jet airplane, turning a corner in a car, riding a merry-go-round, and the circular motion of a tropical cyclone have in common? Each exhibits fictitious forces—unreal forces that arise from motion and may seem real, because the observer’s frame of reference is accelerating or rotating.

When taking off in a jet, most people would agree it feels as if you are being pushed back into the seat as the airplane accelerates down the runway. Yet a physicist would say that you tend to remain stationary while the seat pushes forward on you, and there is no real force backward on you. An even more common experience occurs when you make a tight curve in your car—say, to the right. You feel as if you are thrown (that is, forced) toward the left relative to the car. Again, a physicist would say that you are going in a straight line, but the car moves to the right, and there is no real force on you to the left. Recall Newton’s first law.

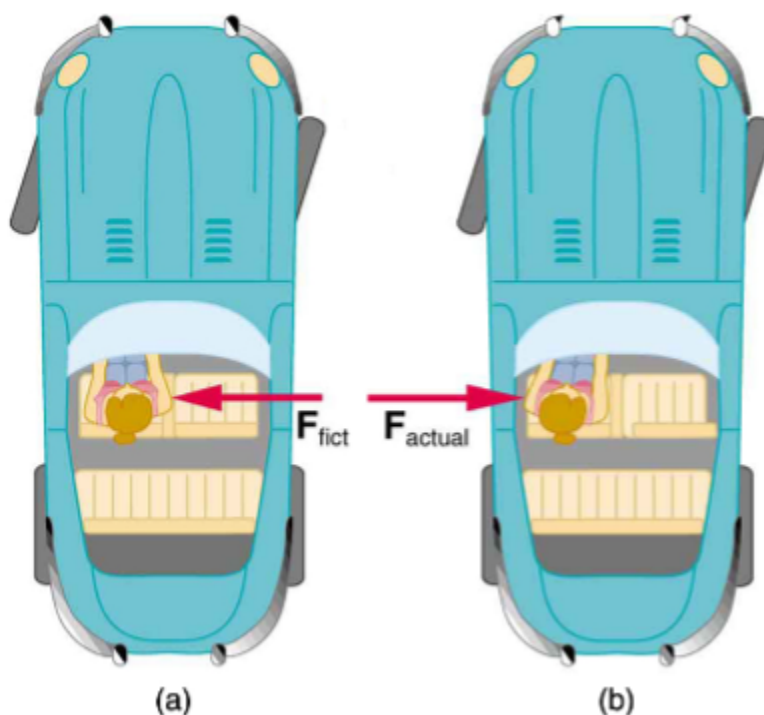


Figure 21: (a) The car driver feels herself forced to the left relative to the car when she makes a right turn. This is a fictitious force arising from the use of the car as a frame of reference. (b) In the Earth's frame of reference, the driver moves in a straight line, obeying Newton's first law, and the car moves to the right. There is no real force to the left on the driver relative to Earth. There is a real force to the right on the car to make it turn.

We can reconcile these points of view by examining the frames of reference used. Let us concentrate on people in a car. Passengers instinctively use the car as a frame of reference, while a physicist uses Earth. The physicist chooses Earth because it is very nearly an inertial frame of reference—one in which all forces are real (that is, in which all forces have an identifiable physical origin). The car is a **non-inertial frame of reference** because it is accelerated to the side. The force to the left sensed by car passengers is a **fictitious force** having no physical origin. There is nothing real pushing them left—the car, as well as the driver, is actually accelerating to the right.

Let us now take a mental ride on a merry-go-round—specifically, a rapidly rotating playground merry-go-round. You take the merry-go-round to be your frame of reference because you rotate together. In that non-inertial frame, you feel a fictitious force, named centrifugal force (not to be confused with centripetal force), trying to throw you off. You must hang on tightly to counteract the **centrifugal force**. In Earth's frame of reference, there is no force trying to throw you off. Rather you must hang on to make yourself go in a circle because otherwise you would go in a straight line, right off the merry-go-round.

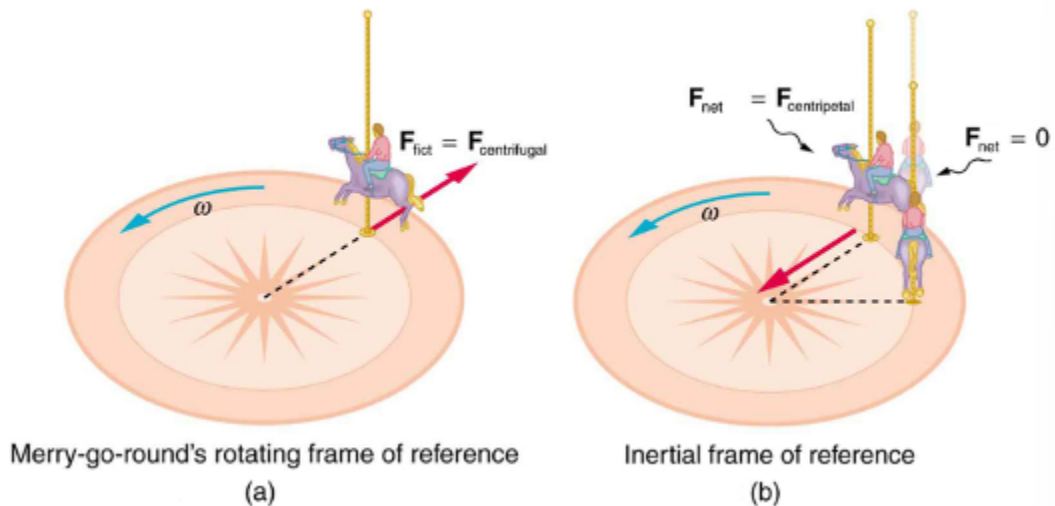


Figure 22: (a) A rider on a merry-go-round feels as if he is being thrown off. This fictitious force is called the centrifugal force—it explains the rider's motion in the rotating frame of reference. (b) In an inertial frame of reference and according to Newton's laws, it is his inertia that carries him off and not a real force (the unshaded rider has $F_{\text{net}} = 0$ and heads in a straight line). A real force, $F_{\text{centripetal}}$, is needed to cause a circular path.

This inertial effect, carrying you away from the center of rotation if there is no centripetal force to cause circular motion, is put to good use in centrifuges (See Figure 23). A centrifuge spins a sample very rapidly. Viewed from the rotating frame of reference, the fictitious centrifugal force throws particles outward, hastening their sedimentation. The greater the angular velocity, the greater the centrifugal force. But what really happens is that the inertia of the particles carries them along a line tangent to the circle while the test tube is forced in a circular path by a centripetal force.

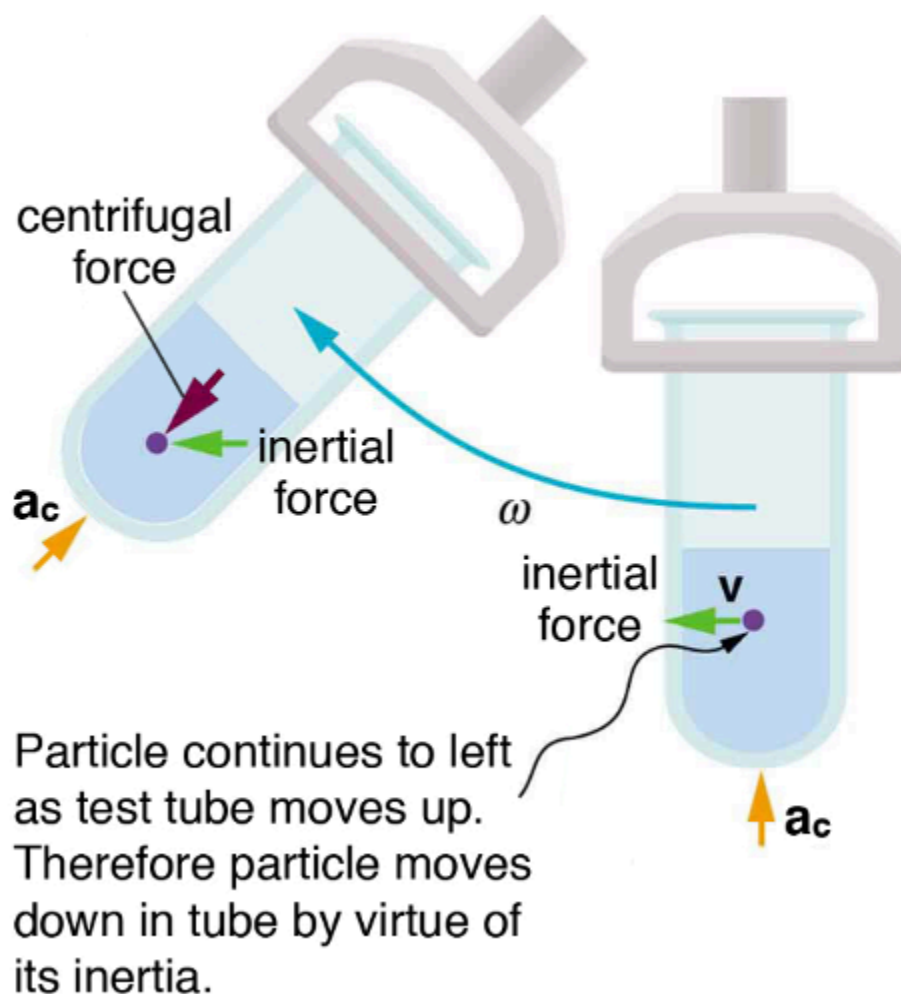


Figure 23: Centrifuges use inertia to perform their task. Particles in the fluid sediment come out because their inertia carries them away from the center of rotation. The large angular velocity of the centrifuge quickens the sedimentation. Ultimately, the particles will come into contact with the test tube walls, which will then supply the centripetal force needed to make them move in a circle of constant radius.

Let us now consider what happens if something moves in a frame of reference that rotates. For example, what if you slide a ball directly away from the center of the merry-go-round, as shown in Figure 24 below? The ball follows a straight path relative to Earth (assuming negligible friction) and a path curved to the right on the merry-go-round's surface. A person standing next to the merry-go-round sees the ball moving straight and the merry-go-round rotating underneath it. In the merry-go-round's frame of reference, we explain the apparent curve to the right by using a fictitious force, called the **Coriolis force**, which causes the ball to curve to the right. The fictitious Coriolis force can be used by anyone in that frame of reference to explain why objects follow curved paths and allows us to apply Newton's Laws in non-inertial frames of reference.

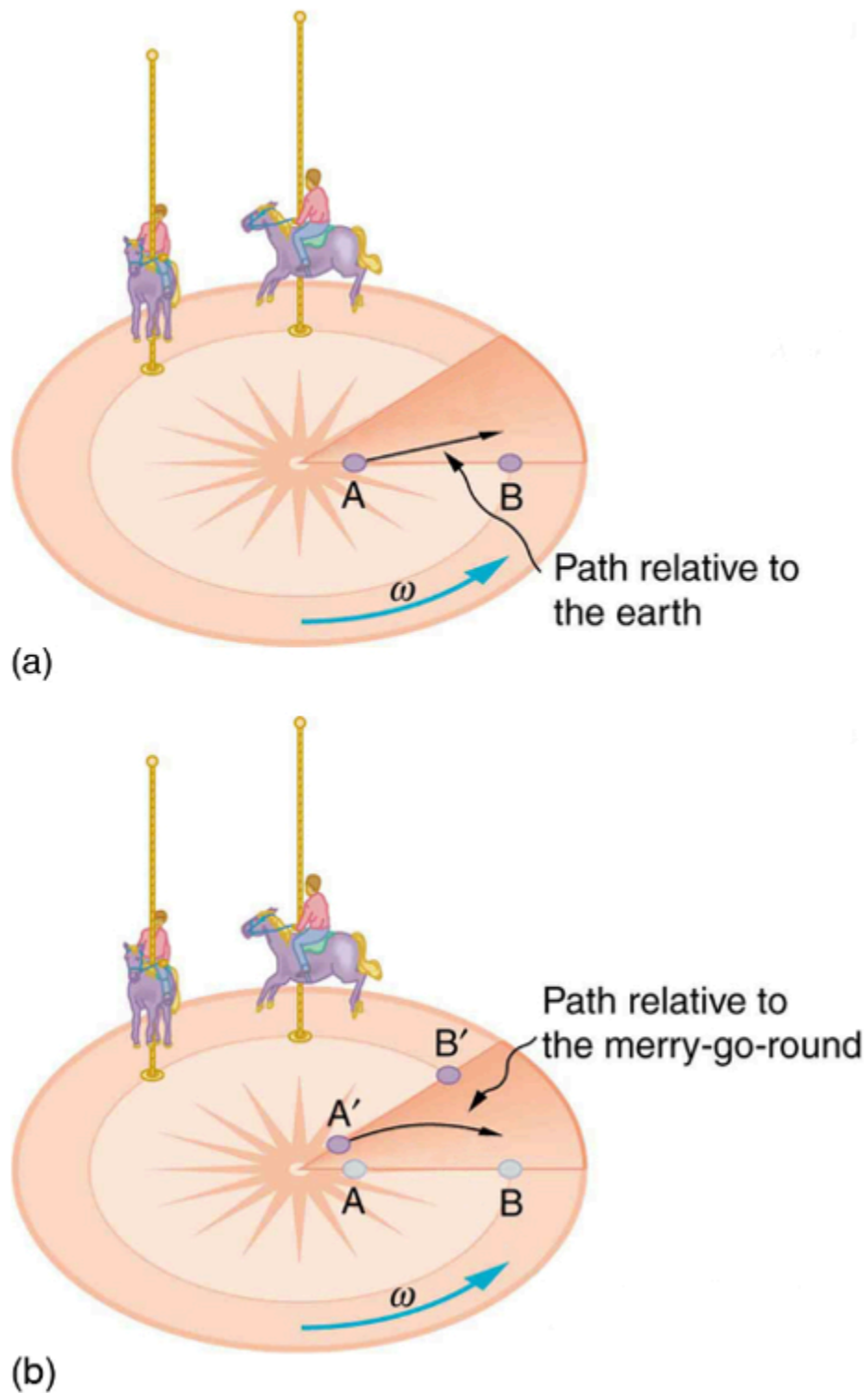


Figure 24: Looking down on the counterclockwise rotation of a merry-go-round, we see that a ball slid straight toward the edge follows a path curved to the right. The person slides the ball toward point B, starting at point A. Both points rotate to the shaded positions (A' and B') shown in the time that the ball follows the curved path in the rotating frame and a straight path in Earth's frame.

RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section explains the relevance of trip distribution, superelevation, stopping sight distance, time and space headways to transportation engineering coursework.

TRIP DISTRIBUTION

In travel demand modeling, one of the major steps (typically Step 2 of the 4-step process) involves estimating the number of trips made from one designated traffic analysis zone (TAZ) to another. This step is termed the trip distribution step. There are multiple trip distribution computational techniques, or “models,” and the gravity model, derived from the physics gravity model (see the above section titled “Laws of Gravitation”), is one of the most useful models. In the physics gravity model, the greater the masses of two objects and the shorter the distance between them, the greater the gravitational pull. In the trip distribution step, the total number of trips produced at the origin zone and attracted at the destination zone are analogous to the object masses. Furthermore, just like the gravity model from Physics, the number of trips is also inversely proportional to an exponent of the distance between zones.

SUPERELEVATION

The above section titled “Circular Motion and Centripetal Acceleration” describes the basics of circular motion. This understanding and the discussion on the balancing of forces in Chapter 6 are critical to the cross-section design of the road segments with horizontal curves.

STOPPING SIGHT DISTANCE

The stopping sight distance, SSD, is the distance needed for a driver to perceive an unusual situation and stop to avoid crashing into an object or another road user. The SSD is estimated based on the time needed for the driver to detect and recognize the situation, also known as perception-reaction time (PRT), and the distance needed to brake to a complete stop, also known as the braking distance. These distances are estimated using the principles of kinematics for motion at constant speeds (see the above section titled “1.1 Describe the Motion of an Object in the Graphic Form through the Time-Space Diagram”), with accelerated motion (see sections “Motion Equations for Constant Acceleration in One Dimension” and “What are the Kinematic Formulas?”), and traveling along a grade (see the section titled “Forces and Newton’s Laws”).

TIME AND SPACE HEADWAYS

The time headway is the time difference between the front bumper of one vehicle and that of another traveling behind it, passing the same point on a roadway segment. Space headway is the physical distance between the front bumper of a vehicle and that of another behind it. The relationship between time and space headways is based on kinematic relationships discussed in the sections titled “Time, Velocity, Speed, and Acceleration” and “Use Kinematic Equations to Solve for Displacement, Time, Velocity and Acceleration of an Object”. The time headway may be used to estimate the traffic flow rate at points of the roadway segments. It is defined as the number of vehicles passing a point on the road per unit of time (in the units of vehicles/

hour). The space headway may be used to estimate the density of the traffic stream on a roadway segment. It is the number of vehicles observed per unit distance on a roadway stretch (vehicles/mile).

Key Takeaways

- The gravity model from physics described in the chapter provides the foundational understanding of the trip distribution modeling between traffic analysis zones (TAZs). The gravity model is based on the idea that the number of trips between two zones is related to the size of the zones (i.e., total trips produced/attracted at a zone based on characteristics such as population or employment) and the distance between them.
- To ensure safe travel, the SSD (Stopping Sight Distance) is estimated based on the design speed appropriate for the roadway segments. Principles of kinematics for motion discussed in this chapter are used to derive formulas for estimating SSD appropriate for the context.
- The idea of motion at constant speed also relates to the time and space headways between vehicles and is useful for estimating flow rate past a point and density on roadway segments.

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- “Traffic Flow” by David Levinson <https://www.youtube.com/watch?v=NjxIhgG4cNw>

CHAPTER 6: BASIC DYNAMICS AND STATIC EQUILIBRIUM

Momentum and collision concepts support an understanding of how speed plays a critical role in the safety of road users, particularly Vulnerable Road Users (VRUs), i.e., pedestrians and bicyclists. Balancing forces on stationary objects is crucial for designing banking or superelevation (slope along the cross-section of the road) of road segments with horizontal curvature.

Learning Objectives

- Balance forces on a stationary object in equilibrium
- Use of momentum preservation to describe the post-collision movement of objects
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter

USE OF MOMENTUM PRESERVATION TO DESCRIBE THE POST-COLLISION MOVEMENT OF OBJECTS

In this section, you will learn about the use of momentum preservation to describe the post-collision movement of objects by watching the videos. Also, examples for your understanding are included.

INTRODUCTION TO MOMENTUM



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Impulse and Momentum Dodgeball Example





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Bouncing Fruit Collision Example



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Momentum: Ice Skater Throws a Ball



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2-Dimensional Momentum Problem (Trig Review!)



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Part 2



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Force vs. Time Graphs



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Check Your Understanding: Introduction to Momentum



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MOMENTUM NOTES – OVERVIEW



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LINEAR MOMENTUM

The scientific definition of linear momentum is consistent with most people's intuitive understanding of momentum: a large, fast-moving object has greater momentum than a smaller, slower object. Linear momentum is defined as the product of a system's mass multiplied by its velocity. In symbols, linear momentum is expressed as $p = mv$

Momentum is directly proportional to the object's mass and also its velocity. Thus, the greater an object's mass or the greater its velocity, the greater its momentum. Momentum p is a vector having the same direction as the velocity v . The SI unit for momentum is $kg \cdot m/s$

Example 1

Calculating Momentum: A Football Player and a Football

(a) Calculate the momentum of a 110-kg football player running at 8.00 m/s.

(b) Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s.

Answers:

$$(a) p_{\text{player}} = (110 \text{ kg})(8.00 \text{ m/s}) = 880 \text{ kg} \cdot \text{m/s}$$

$$(b) p_{\text{ball}} = (0.410 \text{ kg})(25.0 \text{ m/s}) = 10.3 \text{ kg} \cdot \text{m/s}$$

The ratio of the player's momentum to that of the ball is

$$\frac{p_{\text{player}}}{p_{\text{ball}}} = \frac{880}{10.3} = 85.9$$

Momentum and Newton's Second Law

The importance of momentum, unlike the importance of energy, was recognized early in the development of classical physics. Momentum was deemed so important that it was called the “quantity of motion.” Newton actually stated his second law of motion in terms of momentum: The net external force equals the change in momentum of a system divided by the time over which it changes. Using symbols, this law is

$$F_{\text{net}} = \frac{\Delta p}{\Delta t}$$

Where F_{net} is the net external force, Δp is the change in momentum, and Δt is the change in time.

This statement of Newton's Second Law of Motion includes the more familiar $F_{\text{net}} = ma$ as a special case. We can derive this form as follows. First, note that the change in momentum Δp is given by

$$\Delta p = \Delta(mv)$$

If the mass of the system is constant, then

$$\Delta(mv) = m\Delta v$$

So that for constant mass, Newton's Second Law of Motion becomes

$$F_{\text{net}} = \frac{\Delta p}{\Delta t} = \frac{m\Delta v}{\Delta t}$$

Because $\frac{\Delta v}{\Delta t} = a$, we get the familiar equation $F_{\text{net}} = ma$ when the mass of the system is constant.

Example 2

During the 2007 French Open, Venus Williams hit the fastest recorded serve in a premier women's match, reaching a speed of 58 m/s (209 km/h). What is the average force exerted on the 0.057-kg tennis ball by Venus Williams' racquet, assuming that the ball's speed just after impact is 58 m/s, that the initial horizontal component of the velocity before impact is negligible, and that the ball remained in contact with the racquet for 5.0 ms (milliseconds)?

Answer:

$$\begin{aligned}\Delta p &= m(v_f - v_i) \\ &= (0.057 \text{ kg})(58 \text{ m/s} - 0 \text{ m/s}) \\ &= 3.306 \text{ kg} \cdot \text{m/s} \approx 3.3 \text{ kg} \cdot \text{m/s}\end{aligned}$$

Now the magnitude of the net external force can be determined by using

$$\begin{aligned}F_{\text{net}} &= \frac{\Delta p}{\Delta t} \\ F_{\text{net}} &= \frac{\Delta p}{\Delta t} = \frac{3.306 \text{ kg} \cdot \text{m/s}}{5.0 \times 10^{-3} \text{ s}} = 661 \text{ N} \approx 660 \text{ N}\end{aligned}$$

Impulse

The effect of a force on an object depends on how long it acts, as well as how great the force is. In the Example 2 above, a large force acting for a brief time had a significant effect on the momentum of the tennis ball. A small force could cause the same **change in momentum**, but it would have to act for a much longer time. For example, if the ball were thrown upward, the gravitational force (which is much smaller than the tennis racquet's force) would eventually reverse the momentum of the ball. Quantitatively, the effect we are talking about is the change in momentum Δp .

By rearranging the equation $F_{\text{net}} = \frac{\Delta p}{\Delta t}$ to be

$$\Delta p = F_{\text{net}} \Delta t$$

We can see how the change in momentum equals the average net external force multiplied by the time this force acts. The quantity $F_{\text{net}} \Delta t$ is given the name **impulse**. Impulse is the same as the change in momentum.

Example 3

1. Making connections – illustrations of force exerted:

A 1.2-kg block slides across a horizontal, frictionless surface with a constant speed of 3.0 m/s before striking a

fixed barrier and coming to a stop. In Figure 1, the force exerted by the barrier is assumed to be a constant 15 N during the 0.24-s collision. The impulse can be calculated using the area under the curve.

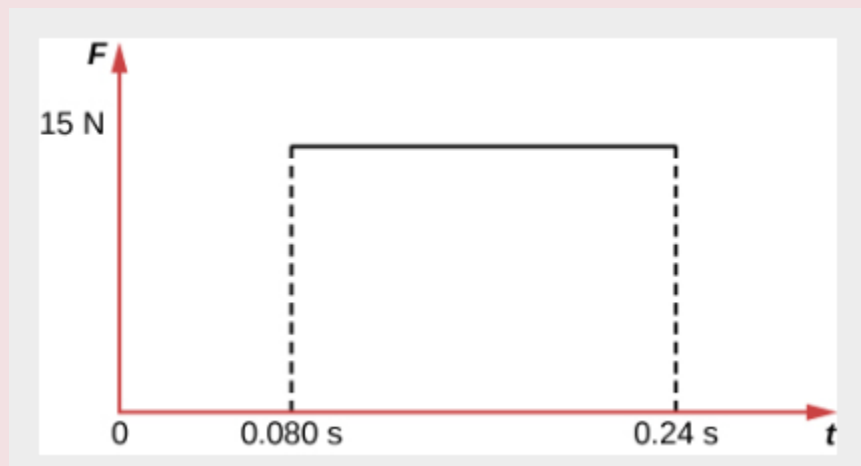


Figure 1: This is a graph showing the force exerted by a fixed barrier on a block versus time.

$$\Delta p = F\Delta t = (15\text{ N})(0.24\text{ s}) = 3.6\text{ kg} \cdot \text{m/s}$$

Note that the initial momentum of the block is

$$p_{\text{initial}} = mv_{\text{initial}} = (1.2\text{ kg})\left(-\frac{3.0\text{ m}}{\text{s}}\right) = -3.6\text{ kg} \cdot \text{m/s}$$

We are assuming that the initial velocity is -3.0 m/s . We have established that the force exerted by the barrier is in the positive direction, so the initial velocity of the block must be in the negative direction. Since the final momentum of the block is zero, the impulse is equal to the change in momentum of the block.

Suppose that, instead of striking a fixed barrier, the block is instead stopped by a spring. Consider the force exerted by the spring over the time interval from the beginning of the collision until the block comes to rest.

In this case, the impulse can be calculated again using the area under the curve (the area of a triangle):

$$\Delta p = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(0.24\text{ s})(30\text{ N}) = 3.6\text{ kg} \cdot \text{m/s}$$

Again, this is equal to the difference between the initial and final momentum of the block, so the impulse is equal to the change in momentum.

2. Calculating Magnitudes of Impulses – Two Billiard Balls Striking a Rigid Wall:

Two identical billiard balls strike a rigid wall with the same speed and are reflected without any change of speed. The first strikes perpendicular to the wall. The second ball strikes the wall at an angle of 30° from the perpendicular and bounces off at an angle of 30° from perpendicular to the wall.

(a) Determine the direction of force on the wall due to each ball.

(b) Calculate the ratio of the magnitudes of impulses on the two balls by the wall.

Answer:

Strategy for A: In order to determine the force on the wall, consider the force on the ball due to the wall using Newton's second law and then apply Newton's third law to determine the direction. Assume the x-axis to be normal to the wall and to be positive in the initial direction of motion. Choose the y-axis to be along the wall in the plane of the second ball's motion. The momentum direction and the velocity direction are the same.

Solution for A: The first ball bounces directly into the wall and exerts a force on it in the +x direction. Therefore, the wall exerts a force on the ball in the -x direction. The second ball continues with the same momentum component in the y direction, but reverses its x-component of momentum, as seen by sketching a diagram of the angles involved and keeping in mind the proportionality between velocity and momentum.

These changes mean the change in momentum for both balls are in the -x direction, so the force of the wall on each ball is along the -x direction.

Strategy for B: Calculate the change in momentum for each ball, which is equal to the impulse imparted to the ball.

Solution for B: Let \diamond be the speed of each ball before and after collision with the wall, and \diamond the mass of each ball. Choose the x-axis and y-axis as previously described and consider the change in momentum of the first ball which strikes perpendicular to the wall.

$$p_{xi} = mu; p_{yi} = 0$$

$$p_{xf} = -mu; p_{yf} = 0$$

Impulse is the change in momentum vector. Therefore, the x-component of impulse is equal to $-2\diamond\diamond$ and the y-component of impulse is equal to zero.

Now consider the change in momentum of the second ball.

$$p_{xi} = mu \cos 30^\circ; p_{yi} = -mu \sin 30^\circ$$

$$p_{xf} = -mu \cos 30^\circ; p_{yf} = -mu \sin 30^\circ$$

It should be noted here that while p_x changes sign after the collision, p_y does not. Therefore, the x-component of impulse is equal to $-2mu \cos 30^\circ$ and the y-component of impulse is equal to zero. The ratio of the magnitudes of the impulse imparted to the balls is $\frac{2mu}{2mu \cos 30^\circ} = \frac{2}{\sqrt{3}} = 1.155$

3. Making connections - baseball:

In most real-life collisions, the forces acting on an object are not constant. For example, when a bat strikes a baseball, the force is small at the beginning of the collision since only a small portion of the ball is initially in contact with the bat. As the collision continues, the ball deforms so that a greater fraction of the ball is in contact with the bat, resulting in a greater force. As the ball begins to leave the bat, the force drops to zero. Although the changing force is difficult to precisely calculate at each instant, the average force can be estimated very well in most cases.

Suppose that a 150-g baseball experiences an average force of 480 N in a direction opposite the initial 32 m/s speed of the baseball over a time interval of 0.017 s. What is the final velocity of the baseball after the collision?

$$\Delta p = F\Delta t = (480)(0.017) = 8.16 \text{ kg} \cdot \text{m/s}$$

$$mv_f - mv_i = 8.16 \text{ kg} \cdot \text{m/s}$$

$$(0.150 \text{ kg})v_f - (0.150 \text{ kg})(-32 \text{ m/s}) = 8.16 \text{ kg} \cdot \text{m/s}$$

$$v_f = 22 \text{ m/s}$$

Conservation of Momentum

Momentum is an important quantity because it is conserved. Yet it was not conserved in the examples in Impulse and Linear Momentum and Force, where large changes in momentum were produced by forces acting on the system of interest. Under what circumstances is momentum conserved?

The answer to this question entails considering a sufficiently large system. It is always possible to find a larger system in which total momentum is constant, even if momentum changes for components of the system. If a football player runs into the goalpost in the end zone, there will be a force on him that causes him to bounce backward. However, the Earth also recoils —conserving momentum—because of the force applied to it through the goalpost. Because Earth is many orders of magnitude more massive than the player, its recoil is immeasurably small and can be neglected in any practical sense, but it is real, nevertheless.

Consider what happens if the masses of two colliding objects are more similar than the masses of a football player and Earth – for example, one car bumping into another, as shown in Figure 2. Both cars are moving in the same direction when the lead car (labeled m_2) is bumped by the trailing car (labeled m_1). The only unbalanced force on each car is the force of the collision. (Assume that the effects due to friction are negligible). Car 1 slows down as a result of the collision, losing some momentum, while car 2 speeds up and gains some momentum. We shall now show that the total momentum of the two-car system remains constant.

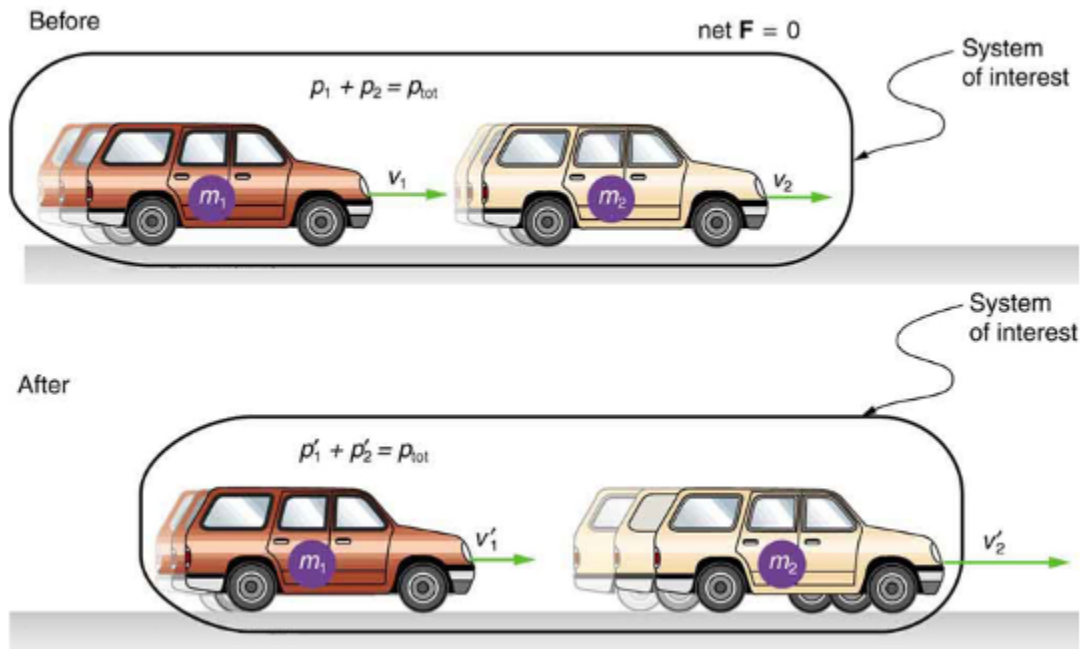


Figure 2: A car of mass m_1 moving with a velocity of v_1 bumps into another car of mass m_2 and velocity v_2 that it is following. As a result, the first car slows down to a velocity of v_1' and the second speeds up to a velocity of v_2' . The momentum of each car is changed, but the total momentum p_{tot} of the two cars is the same before and after the collision (if you assume friction to be negligible)

Using the definition of impulse, the change in momentum of car 1 is given by

$$\Delta p_1 = F_1 \Delta t,$$

where F_1 is the force on car 1 due to car 2, and Δt is the time the force acts (the duration of the collision). Intuitively, it seems obvious that the collision time is the same for both cars, but it is only true for objects traveling at ordinary speeds. This assumption must be modified for objects travelling near the speed of light, without affecting the result that momentum is conserved.

Similarly, the change in momentum of car 2 is

$$\Delta p_2 = F_2 \Delta t,$$

Where F_2 is the force on car 2 due to car 1, and we assume the duration of the collision Δt is the same for both cars. We know from Newton's third law that $F_2 = -F_1$, and so

$$\Delta p_2 = -F_1 \Delta t = -\Delta p_1$$

Thus, the changes in momentum are equal and opposite, and

$$\Delta p_1 + \Delta p_2 = 0$$

Because the changes in momentum add to zero, the total momentum of the two-car system is constant. That is,

$$p_1 + p_2 = \text{constant},$$

$$p_1 + p_2 = p'_1 + p'_2$$

Where p'_1 and p'_2 are the momenta of cars 1 and 2 after the collision. (We often use primes to denote the final state.)

This result—that momentum is conserved—has validity far beyond the preceding one-dimensional case. It can be similarly shown that total momentum is conserved for any isolated system, with any number of objects in it. In equation form, the **conservation of momentum principle** for an isolated system is written

$$p_{tot} = \text{constant}$$

Or

$$p_{tot} = p'_{tot}$$

Where p_{tot} is the total momentum (the sum of the momenta of the individual objects in the system) and p'_{tot} is the total momentum some time later. (The total momentum can be shown to be the momentum of the center of mass of the system.) An **isolated system** is defined to be one for which the net external force is zero ($F_{\text{net}} = 0$)

ELASTIC AND INELASTIC COLLISIONS

Elastic Collisions in One Dimension

Let us consider several types of two-object collisions. These collisions are the easiest to analyze, and they illustrate many of the physical principles involved in collisions. The conservation of momentum principle is particularly useful here, and it can be used whenever the net external force on a system is zero.

We start with the elastic collision of two objects moving along the same line—a one-dimensional problem. An **elastic collision** is one that also conserves internal kinetic energy. **Internal kinetic energy** is the sum of the kinetic energies of the objects in the system. Figure 3 below illustrates an elastic collision in which internal kinetic energy and momentum are conserved.

Truly elastic collisions can only be achieved with subatomic particles, such as electrons striking nuclei. Macroscopic collisions can be very nearly, but not quite, elastic—some kinetic energy is always converted into other forms of energy such as heat transfer due to friction and sound. One macroscopic collision that is nearly elastic is that of two steel blocks on ice. Another nearly elastic collision is that between two carts with spring bumpers on an air track. Icy surfaces and air tracks are nearly frictionless, more readily allowing nearly elastic collisions on them.

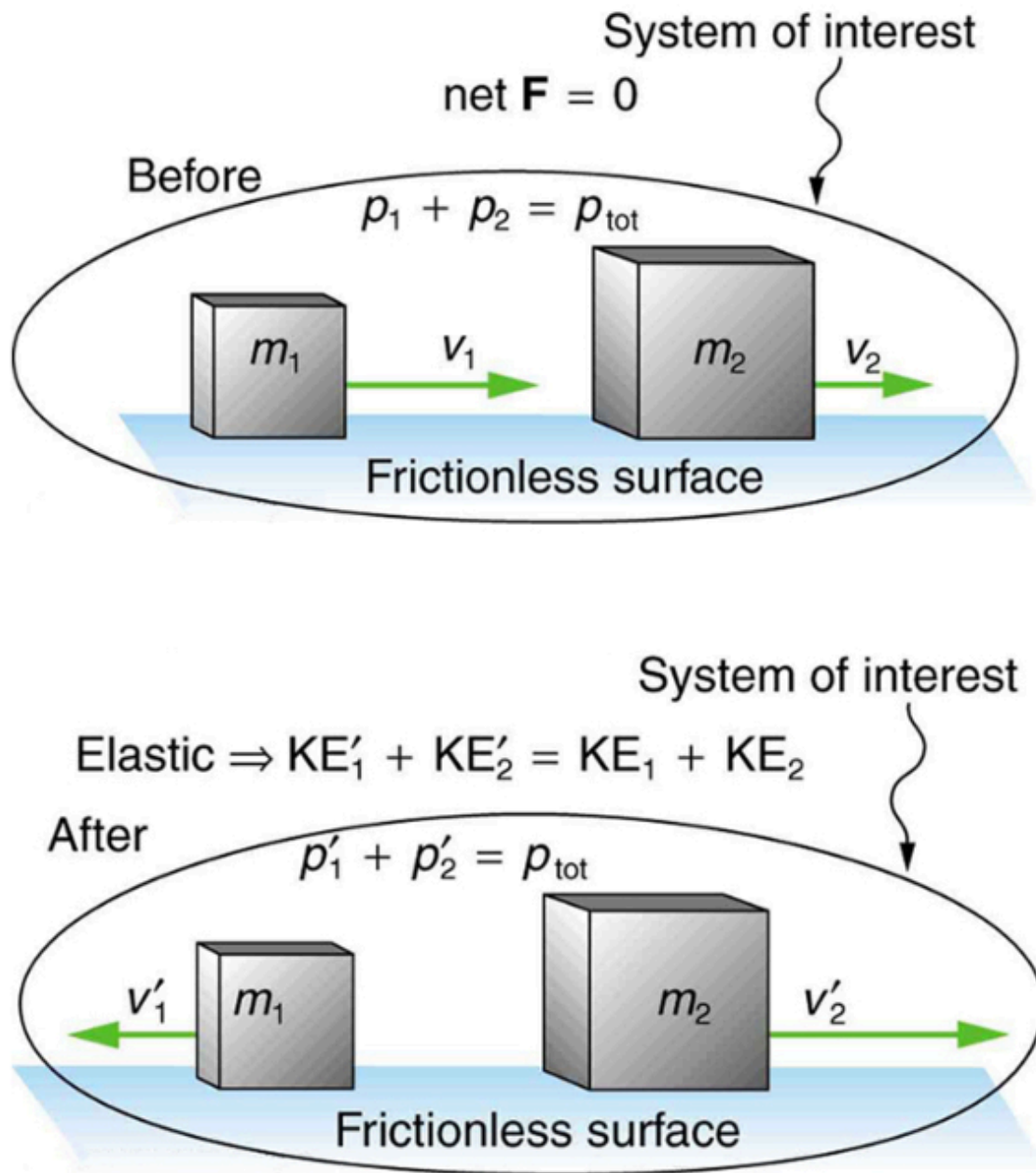


Figure 3: An elastic one-dimensional two-object collision. Momentum and internal kinetic energy are conserved.

Now, to solve problems involving one-dimensional elastic collisions between two objects we can use the equations for conservation of momentum and conservation of internal kinetic energy. First, the equation for conservation of momentum for two objects in a one-dimensional collision is

$$p_1 + p_2 = p'_1 + p'_2 \quad (F_{\text{net}} = 0)$$

Or

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2 \quad (F_{\text{net}} = 0)$$

Where the primes (') indicate values after the collision. By definition, an elastic collision conserves internal kinetic energy, and so the sum of kinetic energies before the collision equals the sum after the collision. Thus,

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \text{ (two - object elastic collision)}$$

Expresses the equation for conservation of internal kinetic energy in a one-dimensional collision.

Example 4

Calculating velocities following an elastic collision:

Calculate the velocities of two objects following an elastic collision, given that

$$m_1 = 0.500 \text{ kg}, m_2 = 3.50 \text{ kg}, v_1 = 4.00 \frac{\text{m}}{\text{s}}, v_2 = 0.$$

Answer: For this problem, note that $v_2 = 0$ and use conservation of momentum.

Thus, $p_1 = p_1' + p_2'$ or $m_1v_1 = m_1v_1' + m_2v_2'$

Using conservation of internal kinetic energy and that $v_2 = 0$,

$$\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$$

Solving the first equation (momentum equation) for v_2' , we obtain

$$v_2' = \frac{m_1}{m_2} (v_1 - v_1').$$

Substituting this expression into the second equation (internal kinetic energy equation) eliminates the variable v_2' , leaving only v_1' as an unknown. There are two solutions to any quadratic equation; in this example, they are $v_1' = 4.00 \text{ m/s}$ and $v_1' = -3.00 \text{ m/s}$. Both solutions may or may not be meaningful. In this case, the first solution is the same as the initial condition. The first solution thus represents the situation before the collision and is discarded. The second solution ($v_1' = -3.00 \text{ m/s}$) is negative, meaning that the first object bounces backward. When this negative value of v_1' is used to find the velocity of the second object after the collision, we get

$$v_2' = \frac{m_1}{m_2} (v_1 - v_1') = \frac{0.500 \text{ kg}}{3.50 \text{ kg}} [4.00 - (-3.00)] \text{ m/s}$$

$$v_2' = 1.00 \text{ m/s}$$

Collision Lab – PhET Simulation





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Inelastic Collisions in One Dimension

We have seen that in an elastic collision, internal kinetic energy is conserved. An **inelastic collision** is one in which the internal kinetic energy changes (it is not conserved). This lack of conservation means that the forces between colliding objects may remove or add internal kinetic energy. Work done by internal forces may change the forms of energy within a system. For inelastic collisions, such as when colliding objects stick together, this internal work may transform some internal kinetic energy into heat transfer. Or it may convert stored energy into internal kinetic energy, such as when exploding bolts separate a satellite from its launch vehicle.

Figure 4 below shows an example of an inelastic collision. Two objects that have equal masses head toward one another at equal speeds and then stick together. Their total internal kinetic energy is initially $\frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2$. The two objects come to rest after sticking together, conserving momentum. But the internal kinetic energy is zero after the collision. A collision in which the objects stick together is something called a **perfectly inelastic collision** because it reduces internal kinetic energy more than does any other type of inelastic collision. In fact, such a collision reduces internal kinetic energy to the minimum it can have while still conserving momentum.

Perfectly inelastic collision – a collision in which the objects stick together is sometimes called “perfectly inelastic.”

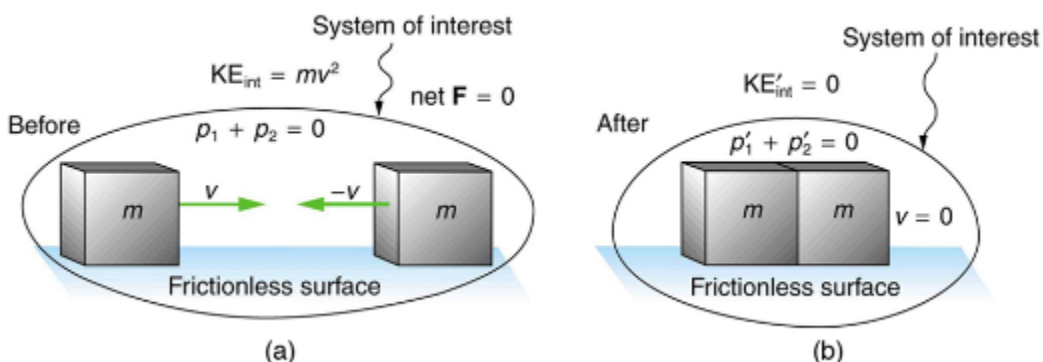


Figure 4: An inelastic one-dimensional two-object collision. Momentum is conserved, but internal kinetic energy is not conserved. (a) Two objects of equal mass initially head directly toward one another at the same speed. (b) The objects stick together (a perfectly inelastic collision), and so their final velocity is zero. The internal kinetic energy of the system changes in any inelastic collision and is reduced to zero in this example.

Example 5

Calculating final velocity and energy release – two carts collide:

In the collision pictured below in Figure 5, two carts collide inelastically. Cart 1 (denoted m_1) carries a spring which is initially compressed. During the collision, the spring releases its potential energy and converts it to internal kinetic energy. The mass of cart 1 and the spring is 0.350 kg, and the cart and the spring together have an initial velocity of 2.00 m/s. Cart 2 (denoted m_2) has a mass of 0.500 kg and an initial velocity of -0.500 m/s. After the collision, cart 1 is observed to recoil with a velocity of -4.00 m/s.

- What is the final velocity of cart 2?
- How much energy was released by the spring (assuming all of it was converted into internal kinetic energy)?

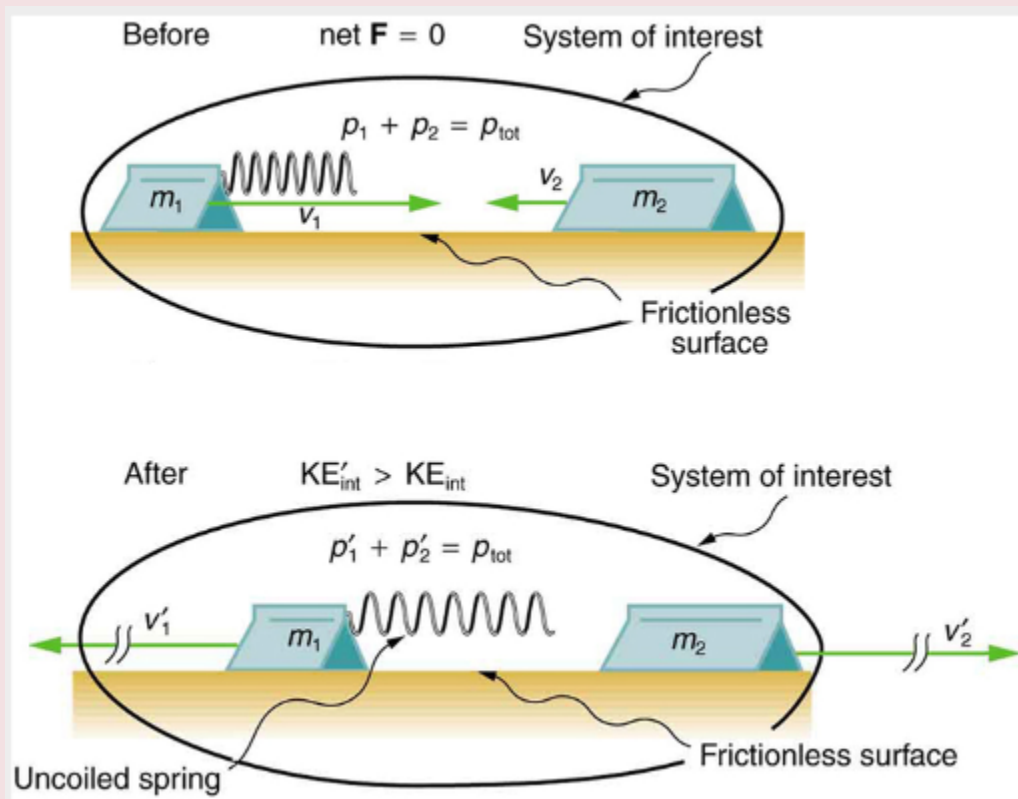


Figure 5: An air track is nearly frictionless, so that momentum is conserved. Motion is one-dimensional. In this collision, examined in the Example above, the potential energy of a compressed spring is released during the collision and is converted to internal kinetic energy.

Answer:

(a) the equation for conservation of momentum in a two-object system is

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2.$$

The only unknown in this equation is v'_2 . Solving for v'_2 and substituting known values into the previous equation yields.

$$\begin{aligned}
 v_2' &= \frac{m_1 v_1 + m_2 v_2 - m_1 v_1'}{m_2} \\
 &= \frac{(0.350 \text{ kg})(2.00 \text{ m/s}) + (0.500 \text{ kg})(-0.500 \text{ m/s})}{0.500 \text{ kg}} - \frac{(0.350 \text{ kg})(-4.00 \text{ m/s})}{0.500 \text{ kg}} \\
 &= 3.70 \text{ m/s}
 \end{aligned}$$

(b) the internal kinetic energy before the collision is

$$\begin{aligned}
 KE_{int} &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \\
 &= \frac{1}{2}(0.350 \text{ kg})(2.00 \text{ m/s})^2 + \frac{1}{2}(0.500 \text{ kg})(-0.500 \text{ m/s})^2 \\
 &= 0.763 \text{ J}
 \end{aligned}$$

After the collision, the internal kinetic energy is

$$\begin{aligned}
 KE'_{int} &= \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 \\
 &= \frac{1}{2}(0.350 \text{ kg})(-4.00 \text{ m/s})^2 + \frac{1}{2}(0.500 \text{ kg})(3.70 \text{ m/s})^2 \\
 &= 6.22 \text{ J}
 \end{aligned}$$

The change in internal kinetic energy is thus

$$KE'_{int} - KE_{int} = 6.22 \text{ J} - 0.763 \text{ J} = 5.46 \text{ J}$$

Collisions of Point Masses in Two Dimensions

In the previous two sections, we considered only one-dimensional collisions; during such collisions, the incoming and outgoing velocities are all along the same line. But what about collisions, such as those between billiard balls, in which objects scatter to the side? These are two-dimensional collisions, and we shall see that their study is an extension of the one-dimensional analysis already presented. The approach taken (similar to the approach in discussing two-dimensional kinematics and dynamics) is to choose a convenient coordinate system and resolve the motion into components along perpendicular axes. Resolving the motion yields a pair of one-dimensional problems to be solved simultaneously.

One complication arising in two-dimensional collisions is that the objects might rotate before or after their collision. For example, if two ice skaters hook arms as they pass by one another, they will spin in circles. To avoid rotation, we consider only the scattering of point masses – that is, structureless particles that cannot rotate or spin.

We start by assuming that $F_{\text{net}} = 0$, so that momentum p is conserved. The simplest collision is one in which one of the particles is initially at rest. See Figure 4 below. The best choice for a coordinate system is one with an axis parallel to the velocity of the incoming particle, as shown in Figure 6. Because momentum is conserved, the

components of momentum along the x- and y- axes (p_x and p_y) will also be conserved, but with the chosen coordinate system, p_y is initially zero and p_x is the momentum of the incoming particle. Both facts simplify the analysis. (Even with the simplifying assumptions of point masses, one particle initially at rest, and a convenient coordinate system, we still gain new insights into nature from the analysis of two-dimensional collisions.)

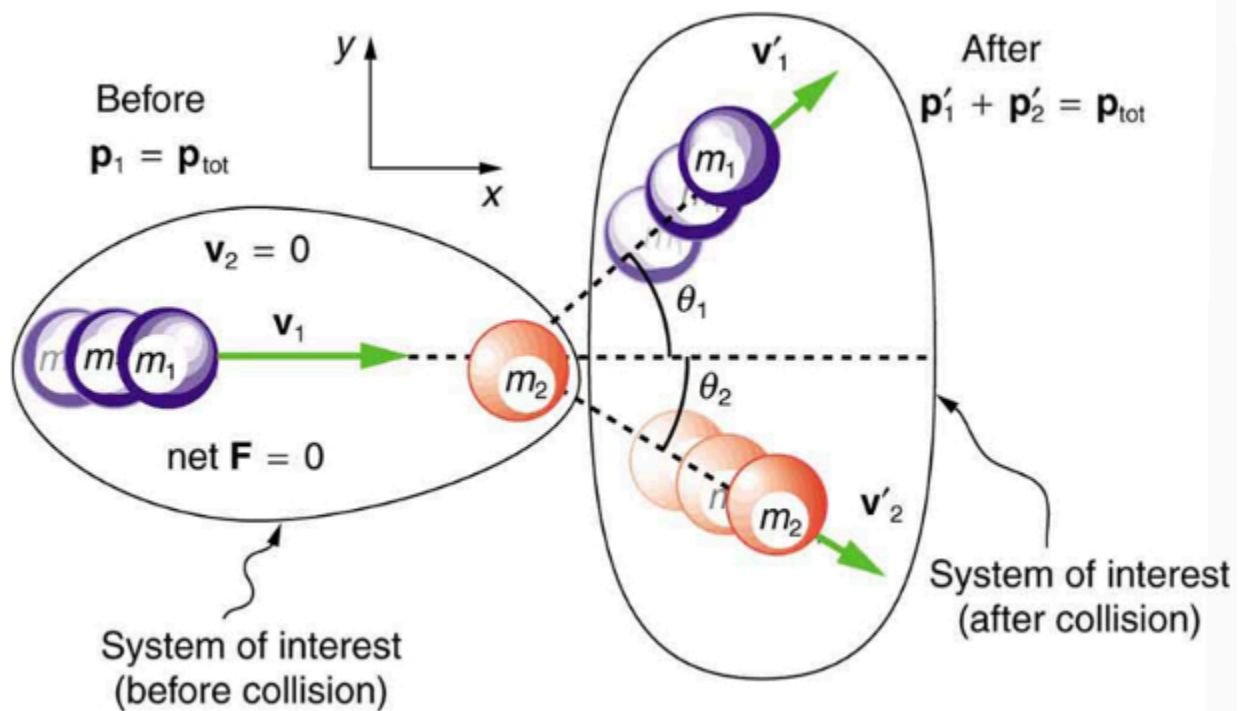


Figure 6: A two-dimensional collision with the coordinate system chosen so that m_2 is initially at rest and v_1 is parallel to the x-axis. This coordinate system is sometimes called the laboratory coordinate system, because many scattering experiments have a target that is stationary in the laboratory, while particles are scattered from it to determine the particles that make-up the target and how they are bound together. The particles may not be observed directly, but their initial and final velocities are.

Along the x-axis, the equation for conservation of momentum is

$$p_{1x} + p_{2x} = p'_{1x} + p'_{2x}$$

Where the subscripts denote the particles and axes, and the primes denote the situation after the collision. In terms of masses and velocities, this equation is

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v'_{1x} + m_2 v'_{2x}$$

But because particle 2 is initially at rest, this equation becomes

$$m_1 v_{1x} = m_1 v'_{1x} + m_2 v'_{2x}$$

The components of the velocities along the x-axis have the form $v \cos \theta$. Because particle 1 initially moves along the x-axis, we find $v_{1x} = v_1$.

Conservation of momentum along the x-axis gives the following equation:

$$m_1 v_{1x} = m_1 v'_{11} \cos \theta_1 + m_2 v'_2 \cos \theta_2$$

Where θ_1 and θ_2 are shown in Figure 4.

Along the y-axis, the equation for conservation of momentum is

$$p_{1y} + p_{2y} = p_{1y} + p_{2y}$$

Or

$$m_1 v_{1y} + m_2 v_{2y} = m_1 v'_{1y} + m_2 v'_{2y}$$

But v_{1y} is zero, because particle 1 initially moves along the x-axis. Because particle 2 is initially at rest, v_{2y} is also zero. The equation for conservation of momentum along the y-axis becomes

$$0 = m_1 v'_{1y} + m_2 v'_{2y}$$

The components of the velocities along the y-axis have the form $v \sin \theta$. Thus, the conservation of momentum along the y-axis gives the following equation:

$$0 = m_1 v'_{11} \sin \theta_1 + m_2 v'_2 \sin \theta_2$$

Making Connections: Real World Connections

We have seen, in one-dimensional collisions when momentum is conserved, that the center-of-mass velocity of the system remains unchanged as a result of the collision. If you calculate the momentum and center-of-mass velocity before the collision, you will get the same answer as if you calculate both quantities after the collision. This logic also works for two-dimensional collisions.

For example, consider two cars of equal mass. Car A is driving east (+x-direction) with a speed of 40 m/s. Car B is driving north (+y-direction) with a speed of 80 m/s. What is the velocity of the center-of-mass of this system before and after an inelastic collision, in which the cars move together as one mass after the collision?

Since both cars have equal mass, the center-of-mass velocity components are just the average of the components of the individual velocities before the collision. The x-component of the center of mass velocity is 20 m/s, and the y-component is 40 m/s.

Using momentum conservation for the collision in both the x-component and y-component yields similar answers:

$$m(40) + m(0) = (2m)v_{\text{final}}(x)$$

$$v_{\text{final}}(x) = 20 \text{ m/s}$$

$$m(0) + m(80) = (2m)v_{\text{final}}(y)$$

$$v_{\text{final}}(y) = 40 \text{ m/s}$$

Since the two masses move together after the collision, the velocity of this combined object is equal to the center-of-mass velocity. Thus, the center-of-mass velocity before and after the collision is identical, even in two-dimensional collisions, when momentum is conserved.

ELASTIC AND INELASTIC COLLISIONS

Elastic and Inelastic Collisions



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Deriving the Shortcut to Solve Elastic Collision Problems



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How to Use the Shortcut for Solving Elastic Collisions



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BALANCE FORCES ON A STATIONARY OBJECT IN EQUILIBRIUM

EQUILIBRIUM FORCES



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Check Your Understanding: Equilibrium Forces



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STATIC EQUILIBRIUM



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Balancing Act PhET Simulation



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THE FIRST CONDITION FOR EQUILIBRIUM

The first condition necessary to achieve equilibrium is that the net external force on the system must be zero. Expressed as an equation, this is simply

$$\text{net } F = 0$$

Note that if net F is zero, then the net external force in *any* direction is zero. For example, the net external forces along the typical x - and y -axes are zero. This is written as net $F_x = 0$ and $F_y = 0$.

THE SECOND CONDITION FOR EQUILIBRIUM

The second condition necessary to achieve equilibrium involves avoiding accelerated rotation (maintaining a constant angular velocity). A rotating body or system can be in equilibrium if its rate of rotation is constant and remains unchanged by the forces acting on it. The magnitude, direction, and point of application of the force are incorporated into the definition of the physical quantity called **torque**.

Torque is the rotational equivalent of a force. It is a measure of the effectiveness of a force in changing or accelerating a rotation (changing the angular velocity over a period of time). In equation form, the magnitude of torque is defined to be

$$\tau = rF \sin(\theta)$$

Where τ is the symbol for torque, r is the distance from the pivot point to the point where the force is applied, F is the magnitude of the force, and θ is the angle between the force and the vector directed from the point of application to the pivot point.

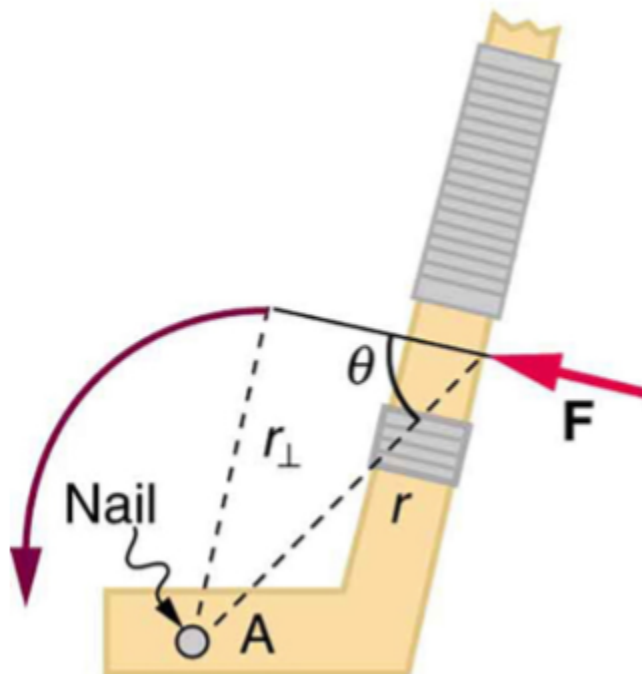


Figure 7: A force applied to an object can produce a torque, which depends on the location of the pivot point. The three factors r , F , and θ for pivot point A on a body are shown here – r is the distance from the chosen pivot point to the point where the force F is applied, and θ is the angle between F and the vector directed from the point of application to the pivot point. If the object can rotate around point A, it will rotate counterclockwise. This means that torque is counterclockwise relative to pivot A.

The perpendicular lever arm r_{\perp} is the shortest distance from the pivot point to the line along which F acts; it is shown as a dashed line in Figure 7. Note that the line segment that defines the distance r_{\perp} is perpendicular to F , as its name implies. It is sometimes easier to find or visualize r_{\perp} than to find both r and θ . In such cases, it may be more convenient to use $\tau = r_{\perp}F$ rather than $\tau = rF \sin(\theta)$ for torque, but both are equally valid.

The SI unit of torque is newtons time meters, usually written as $N \cdot m$. For example, if you push perpendicular to the door with a force of 40 N at a distance of 0.800 m from the hinges, you exert a torque of $32 N \cdot m$ ($0.800m \times 40N \times \sin(90^\circ)$) relative to the hinges. If you reduce the force to 20 N, the torque is reduced to $16 N \cdot m$, and so on.

Now, the second condition necessary to achieve equilibrium is that the net external torque on a system must be zero. An external torque is one that is created by an external force. You can choose the point around which the torque is calculated. The point can be the physical pivot point of a system or any other point in space—but it must be the same point for all torques. If the second condition (net external torque on a system is zero) is satisfied for one choice of pivot point, it will also hold true for any other choice of pivot point in or out of the system of interest. (This is true only in an inertial frame of reference.) The second condition necessary to achieve equilibrium is stated in equation form as $\text{net } \tau = 0$ where net means total. Torques, which are in opposite directions are assigned opposite signs. A common convention is to call counterclockwise (ccw) torques positive and clockwise (cw) torques negative.

When two children balance a seesaw as shown in Figure 8, they satisfy the two conditions for equilibrium. Most people have perfect intuition about seesaws, knowing that the lighter child must sit farther from the pivot and that a heavier child can keep a lighter one off the ground indefinitely.

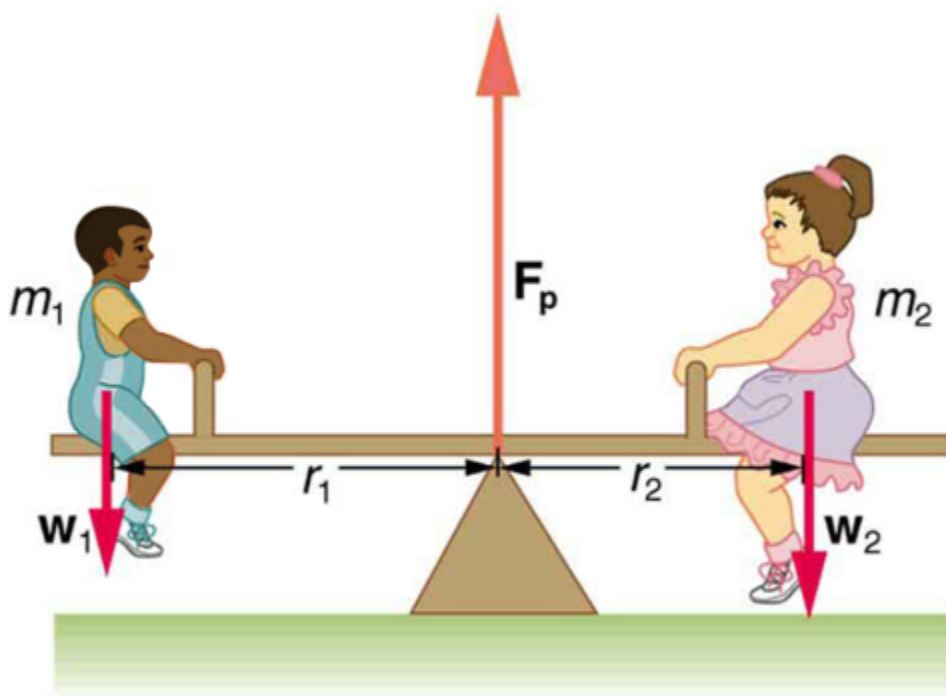


Figure 8: Two children balancing a seesaw satisfy both conditions for equilibrium. The lighter child sits farther from the pivot to create a torque equal in magnitude to that of the heavier child.

Example 6

The two children shown in Figure 8 above are balanced on a seesaw of negligible mass. (This assumption is made to keep the example simple). The first child has a mass of 26.0 kg and sits 1.60 m from the pivot. (a) If the second child has a mass of 32.0 kg, how far is she from the pivot? (b) What is F_p the supporting force exerted by the pivot?

Strategy:

Both conditions for equilibrium must be satisfied. In part (a), we are asked for a distance; thus, the second condition (regarding torques) must be used, since the first (regarding only forces) has no distances in it. To apply the second condition for equilibrium, we first identify the system of interest to be the seesaw plus the two children. We take the supporting pivot to be the point about which the torques are calculated. We then identify all external forces acting on the system.

Solution (a)

The three external forces acting on the system are the weights of the two children and the supporting force of the pivot. Let us examine the torque produced by each. Torque is defined to be

$$\tau = rF \sin(\theta)$$

Here $\theta = 90^\circ$, so that $\sin(\theta) = 1$ for all three forces. That means $\tau_\perp = r$ for all three. The torques exerted by the three forces are first,

$$\tau_1 = r_1 w_1$$

Second,

$$\tau_2 = -r_2 w_2$$

And third,

$$\tau_p = r_p F_p$$

$$= 0 \cdot F_p$$

$$= 0$$

Note that a minus sign has been inserted into the second question because this torque is clockwise and is therefore negative by convention. Since F_p act directly on the pivot point, the distance r_p is zero. A force acting on the pivot cannot cause a rotation, just as pushing directly on the hinges of a door will not cause it to rotate. Now, the second condition for equilibrium is that the sum of the torques on both children is zero. Therefore

$$\tau_2 = -\tau_1$$

Or

$$r_2 w_2 = r_1 w_1$$

Weight is mass times the acceleration due to gravity. Entering mg for w , we get

$$r_2 m_2 g = r_1 m_1 g$$

Solve this for the unknown r_2 :

$$r_2 = r_1 \frac{m_1}{m_2}$$

The quantities on the right side of the equation are known; thus, r_2 is

$$r_2 = (1.60 \text{ m}) \frac{26.0 \text{ kg}}{32.0 \text{ kg}} = 1.30 \text{ m}$$

As expected, the heavier child must sit closer to the pivot (1.30 m versus 1.60 m) to balance the seesaw.

Solution (b)

This part asks for a force F_p . The easiest way to find it is to use the first condition for equilibrium, which is $\text{net } F = 0$

The forces are all vertical, so that means we are dealing with a one-dimension problem along the vertical axis; hence, the condition can be written as

$$\text{net } F_y = 0$$

Where we again call the vertical axis the y-axis. Choosing upward to be the positive direction, and using plus and minus signs to indicate the directions of the forces, we see that

$$F_p - w_1 - w_2 = 0$$

This equation yields what might have been guessed at the beginning:

$$F_p = w_1 + w_2$$

So, the pivot supplies a supporting force equal to the total weight of the system:

$$F_p = m_1g + m_2g$$

Entering known values gives

$$\begin{aligned} F_p &= (26.0 \text{ kg}) (9.80 \text{ m/s}^2) + (32.0 \text{ kg}) (9.80 \text{ m/s}^2) \\ &= 568N \end{aligned}$$

Discussion:

The two results make intuitive sense. The heavier child sits closer to the pivot. The pivot supports the weight of the two children. Part (b) can also be solved using the second condition for equilibrium, since both distances are known, but only if the pivot point is chosen to be somewhere other than the location of the seesaw's actual pivot!

Several aspects of the preceding example have broad implications. First, the choice of the pivot as the point around which torques are calculated simplified the problem. Since F_p is exerted on the pivot point, its lever arm is zero. Hence, the torque exerted by the supporting force F_p is zero relative to that pivot point. The second condition for equilibrium holds for any choice of pivot point, and so we choose the pivot point to simplify the solution of the problem.

Second, the acceleration due to gravity canceled in this problem, and we were left with a ratio of masses. This will not always be the case. Always enter the correct forces – do not jump ahead to enter some ratio of masses.

Third, the weight of each child is distributed over an area of the seesaw, yet we treated the weights as if each

force were exerted at a single point. This is not an approximation – the distances r_1 and r_2 are the distances to points directly below the center of gravity of each child.

Finally, not that the concept of torque has an importance beyond static equilibrium. Torque plays the same role in rotational motion that force plays in linear motion.

STABILITY

A system is said to be in stable equilibrium if, when displaced from equilibrium, it experiences a net force or torque in a direction opposite to the direction of the displacement. For example, a marble at the bottom of a bowl will experience a restoring force when displaced from its equilibrium position. This force moves it back toward the equilibrium position. Most systems are in stable equilibrium, especially for small displacements. For another example of stable equilibrium, see the pencil in Figure 9.

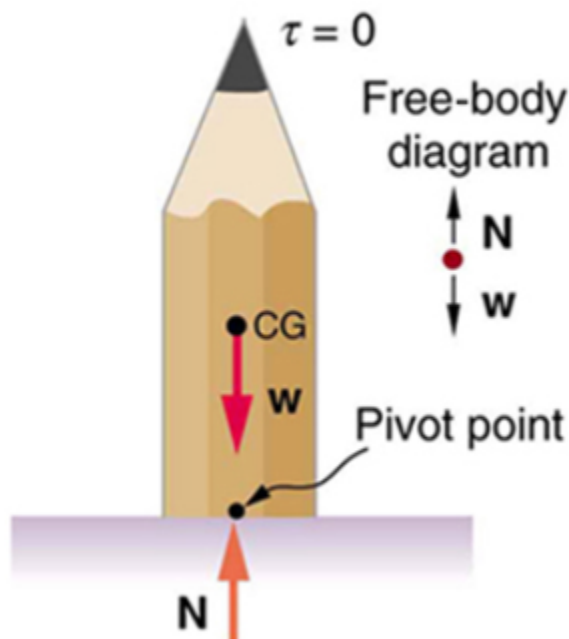


Figure 9: This pencil is in the condition of equilibrium. The net force on the pencil is zero and the total torque about any pivot is zero.

A system is in unstable equilibrium if, when displaced, it experiences a net force or torque in the same direction as the displacement from equilibrium. A system in unstable equilibrium accelerates away from its equilibrium position if displaced even slightly. An obvious example is a ball resting on top of a hill. Once displaced, it accelerates away from the crest. See Figures 10-13 for examples of unstable equilibrium.

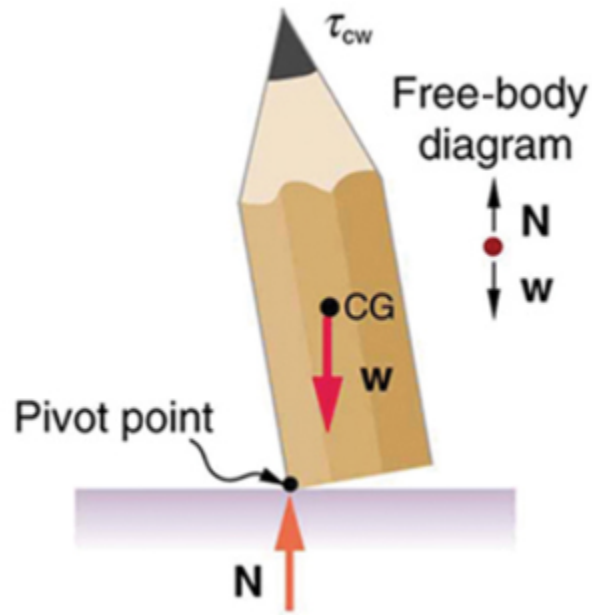


Figure 10: If the pencil is displaced slightly to the side (counterclockwise), it is no longer in equilibrium. Its weight produces a clockwise torque that returns the pencil to its equilibrium position.

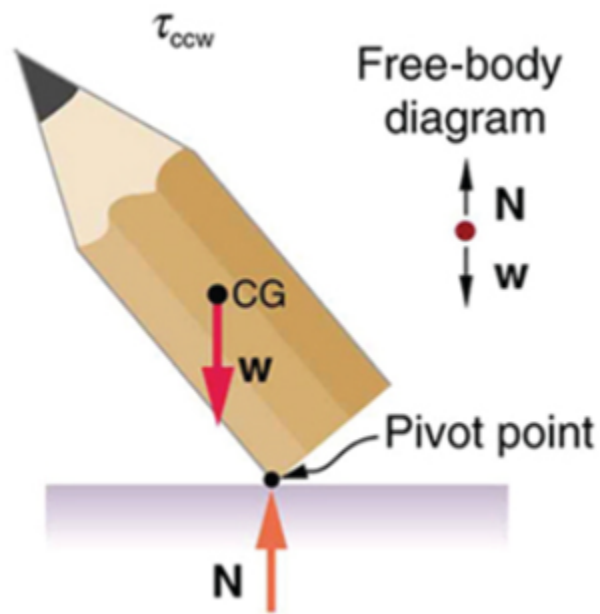


Figure 11: If the pencil is displaced too far, the torque caused by its weight changes to counterclockwise and causes the displacement to increase.

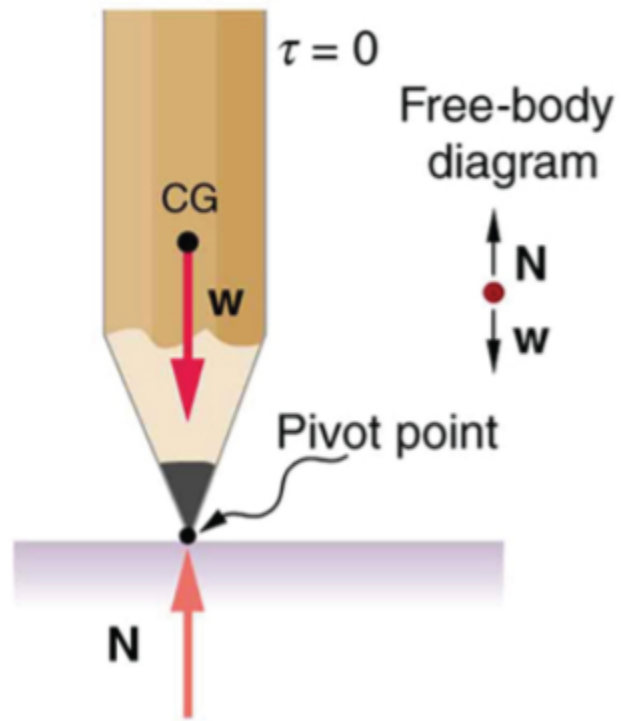


Figure 12: This figure shows unstable equilibrium, although both conditions for equilibrium are satisfied.

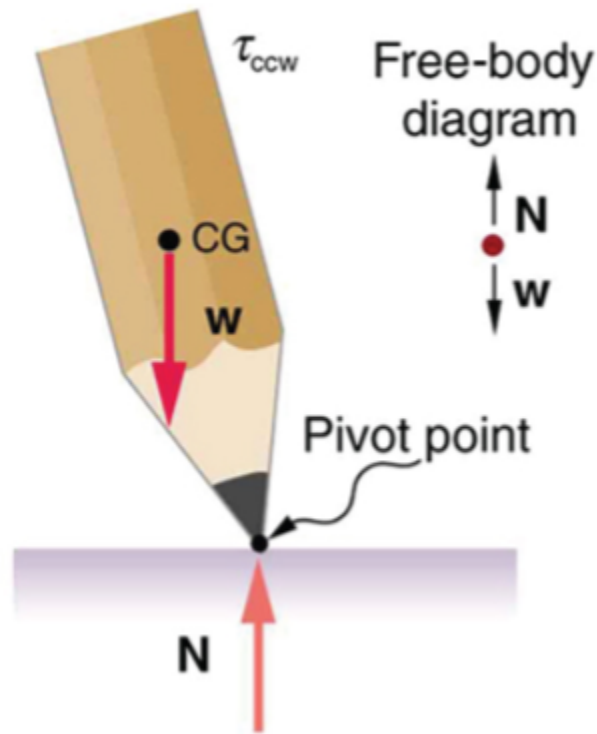


Figure 13: If the pencil is displaced even slightly, a torque is created by its weight that is in the same direction as the displacement, causing the displacement to increase.

A system is in neutral equilibrium if its equilibrium is independent of displacements from its original position. A marble on a flat horizontal surface is an example. Combinations of these situations are possible. For example, a marble on a saddle is stable for displacements toward the front or back of the saddle and unstable for displacements to the side. Figures 14-15 show another example of neutral equilibrium.

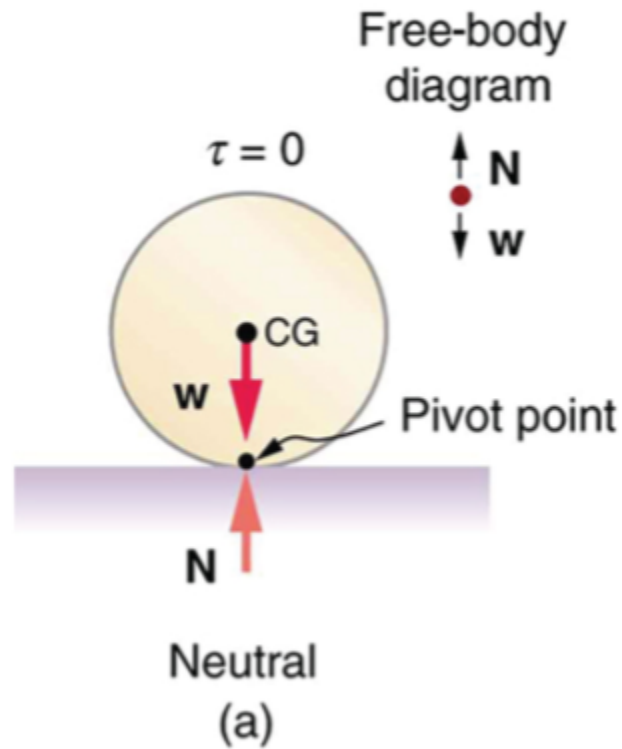


Figure 14: Here we see neutral equilibrium. The cg of a sphere on a flat surface lies directly above the point of support, independent of the position on the surface. The sphere is therefore in equilibrium in any location, and if displaced, it will remain put.

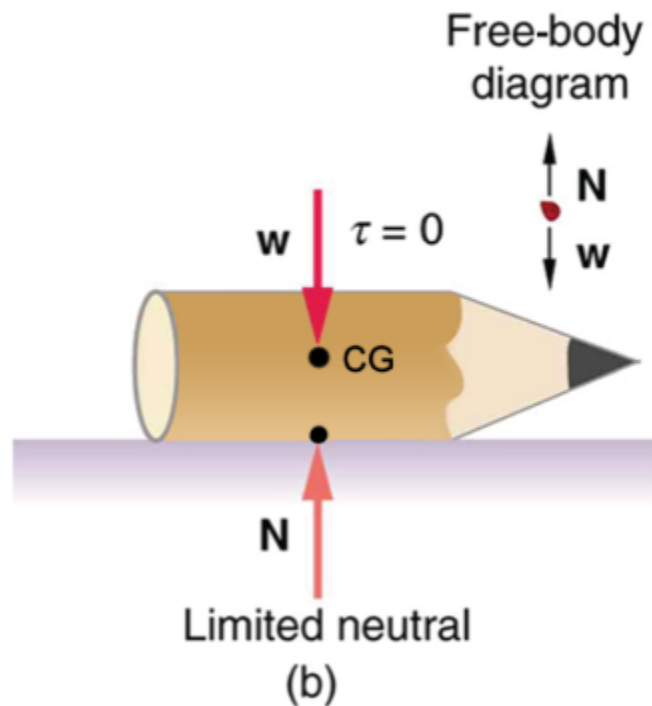


Figure 15: Because it has a circular cross section, the pencil is in neutral equilibrium for displacements perpendicular to its length.

Center of Gravity



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RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section explains the relevance of traffic safety and superelevation to transportation engineering coursework.

TRAFFIC SAFETY

Avoiding collisions (or crashes) and in case collisions do occur minimization of harm to human beings involved are of critical importance in transportation engineering practice. Practitioners and researchers recommend use of term “crashes” or “collisions” instead of “accidents” so that they are thought of as preventable events instead of inevitable incidents with no possible avoidance/mitigation measures. Crashes are classified according to their severity from property-damage-only (PDO) to fatal, and the crash severity is highly correlated with the impact speed, and hence the momentum. Mitigation measures, such as guardrails, traffic signs with breakaway poles, crash attenuators, air bags, seat belts, and child seats are all implemented to reduce the severities of crashes according to the momentum principles, described in the

section titled “Use of Momentum Preservation to Describe the Post-Collision Movement of Objects.”

SUPERELEVATION

Superelevation, i.e., raising of outer edge of the pavement relative to the inner edge, is provided to support higher speed travel along a horizontal curve. The banking helps provide the required inward force to keep vehicle on a circular motion. Figure 14 illustrates a superelevated curved segment of racetrack, where the relative elevation of the outer edge is more pronounced (compared to public roadways) due to significantly higher speeds of race cars. The formula for estimating the required superelevation for a given design speed is based on the principles of balancing forces described in the above section titled “Balance Forces on a Stationary Object in Equilibrium.”



Figure 16: Superelevated segment of a racetrack with raised outer edge

Key Takeaways

- Transportation engineers and safety professionals recognize that crashes and collisions are not “accidents” in the sense that these events are not inevitable, can be prevented, and their impacts can be mitigated. One key aspect of crash impact mitigation is to reduce the severity of collisions that do occur, especially those involving VRUs. The speed-momentum relationships discussed in this chapter are essential to developing this understanding.
- Superelevation or banking refers to the raised outer edge of a roadway segment on a horizontal curve.

It is provided to counteract the tendency of a vehicle to roll outward on a curve. With a gravitational force component pulling the vehicle inward, the superelevation helps keep the vehicle on a circular path providing stability and control for negotiating circular curve segments at higher speeds.

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REFERENCES

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CHAPTER 7: WAVES AND DOPPLER EFFECT

Propagation of sound waves is critical to understanding and mitigating the noise pollution resulting from transportation projects. A related concept, i.e., the Doppler Effect, is helpful in understanding the use of radar guns for traffic speed enforcement and spot speed studies.

Learning Objectives

- Define a wave and its basic properties (e.g., Period, Wavelength, and Frequency)
- List different wave types
- Describe the movement of sound waves
- Describe Doppler Effect and its implications
- Identify topics in the introductory transportation engineering courses that build on the Physics concept discussed in this course

WAVE PROPERTIES

What do we mean when we say something is a wave? The most intuitive and easiest wave to imagine is the familiar water wave. More precisely, a wave is a disturbance that propagates or moves from the place it was created. For water waves, the disturbance is in the surface of the water, perhaps created by a rock thrown into a pond or by a swimmer splashing the surface repeatedly. For sound waves, the disturbance is a change in air pressure, perhaps created by the oscillating cone inside a speaker. For earthquakes, there are several types of disturbances, including disturbance of Earth's surface and pressure disturbances under the surface. Even radio waves are most easily understood using an analogy with water waves. Visualizing water waves is useful because there is more to it than just a mental image. Water waves exhibit characteristics common to all waves, such as amplitude, period, frequency, and energy. All wave characteristics can be described by a small set of underlying principles.

WAVE MOVEMENT

A wave is a disturbance that propagates or moves from the place it was created. The simplest waves repeat themselves for several cycles and are associated with simple harmonic motion. Let us start by considering the simplified water wave in Figure 1 below. The wave is an up

and down disturbance of the water surface. It causes a sea gull to move up and down in simple harmonic motion as the wave crests and troughs (peaks and valleys) pass under the bird. The time for one complete up and down motion is the wave's period \diamond . The wave's frequency $f = \frac{1}{T}$. The wave itself moves to the right in the figure. This movement of the wave is actually the disturbance moving to the right, not the water itself (or the bird would move to the right). We define wave velocity \mathcal{V}_W to be the speed at which the disturbance moves. Wave velocity is sometimes also called the propagation velocity or propagation speed because the disturbance propagates from one location to another.

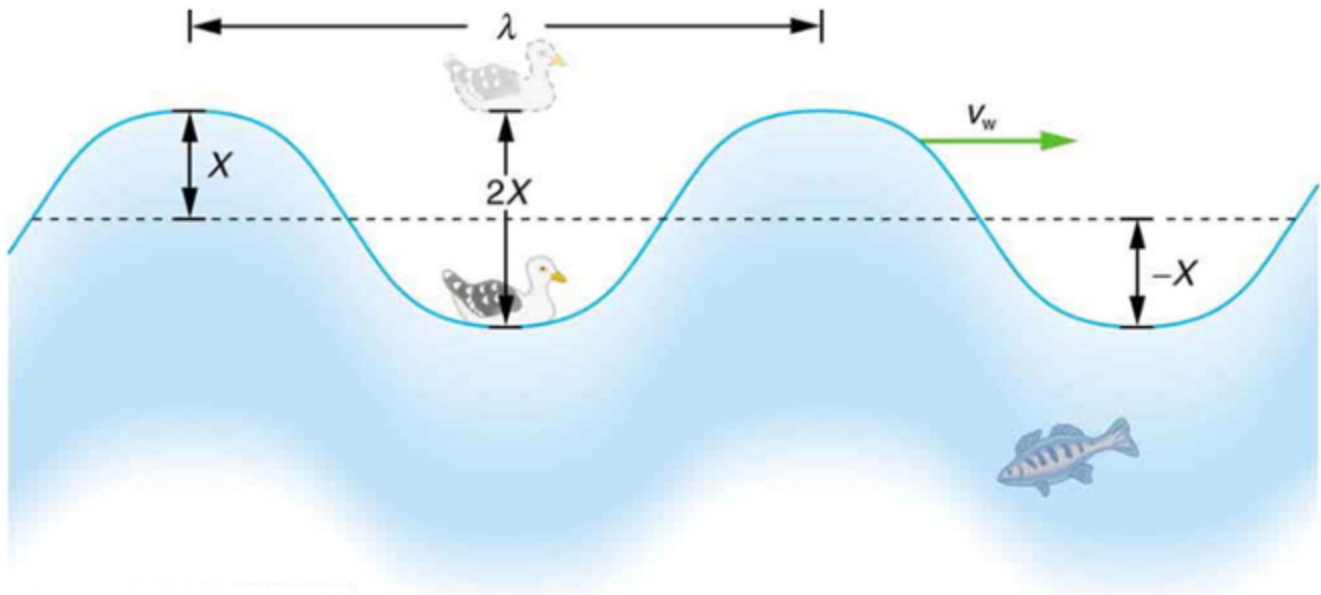


Figure 1: An idealized ocean wave passes under a seagull that bobs up and down in simple harmonic motion. The wave has wavelength λ , which is the distance between adjacent identical parts of the wave. The up and down disturbance of the surface propagates parallel to the surface at a speed \mathcal{V}_W .

The water wave in Figure 1 also has a length associated with it, called its wavelength \diamond , the distance between adjacent identical parts of a wave. (\diamond is the distance parallel to the direction of propagation.) The speed of propagation \mathcal{V}_W is the distance the wave travels in a given time, which is one wavelength in the time of one period. In equation form, that is

$$v_w = \frac{\lambda}{T}$$

Or

$$v_w = f\lambda$$

This fundamental relationship holds for all types of waves. For water waves, \mathcal{V}_W is the speed of a surface wave; for sound, \mathcal{V}_W is the speed of sound; and for visible light, \mathcal{V}_W is the speed of light, for example.

Applying the Science Practices: Different Types of Waves

Consider a spring fixed to a wall with a mass connected to its end. This fixed point on the wall exerts a force on the complete spring-and-mass system, and this implies that the momentum of the complete system is not conserved. Now, consider energy. Since the system is fixed to

a point on the wall, it does not do any work; hence, the total work done is conserved, which means that the energy is conserved. Consequently, we have an oscillator in which energy is conserved but momentum is not. Now, consider a system of two masses connected to each other by a spring. This type of system also forms an oscillator. Since there is no fixed point, momentum is conserved as the forces acting on the two masses are equal and opposite. Energy for such a system will be conserved because there are no external forces acting on the spring-two-masses system. It is clear from above that, for momentum to be conserved, momentum needs to be carried by waves. This is a typical example of a mechanical oscillator producing mechanical waves, which need a medium in which to propagate. Sound waves are also examples of mechanical waves. Some types of waves can travel in the absence of a medium of propagation. Such waves: they are called “electromagnetic waves.” Light waves are examples of electromagnetic waves. Electromagnetic waves are created by the vibration of electric charge. which This vibration creates a wave with both electric and magnetic field components.

Transverse and Longitudinal Waves

A simple wave consists of a periodic disturbance that propagates from one place to another. The wave in Figure 2 below propagates in the horizontal direction while the surface is disturbed in the vertical direction. Such a wave is called a transverse wave or shear wave; in such a wave, the disturbance is perpendicular to the direction of propagation. In contrast, in a longitudinal wave or compressional wave, the disturbance is parallel to the direction of propagation. Figure 3 below shows an example of a longitudinal wave. The size of the disturbance is its amplitude X and is completely independent of the speed of propagation v_w .

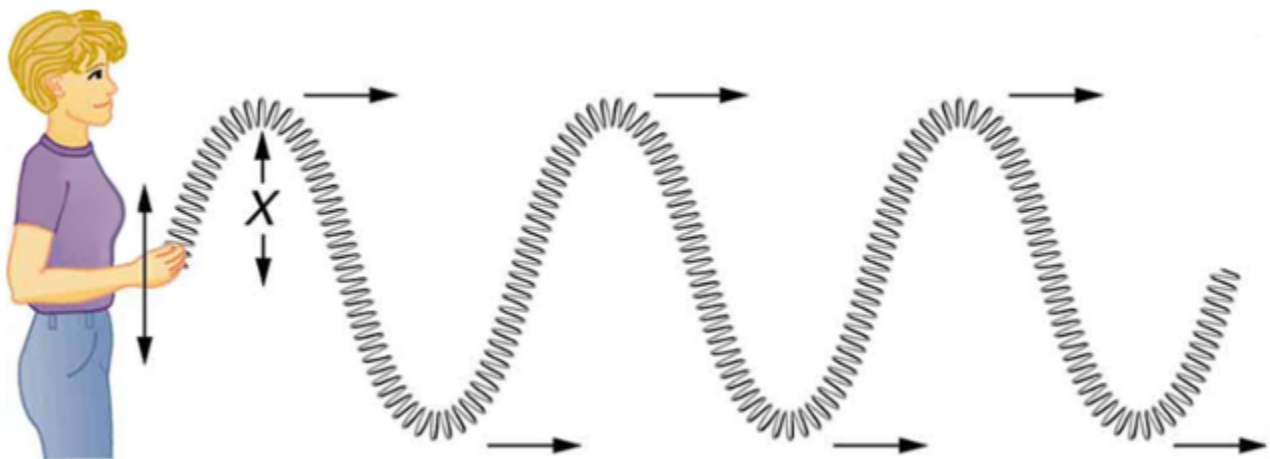


Figure 2: In this example of a transverse wave, the wave propagates horizontally, and the disturbance in the cord is in the vertical direction.

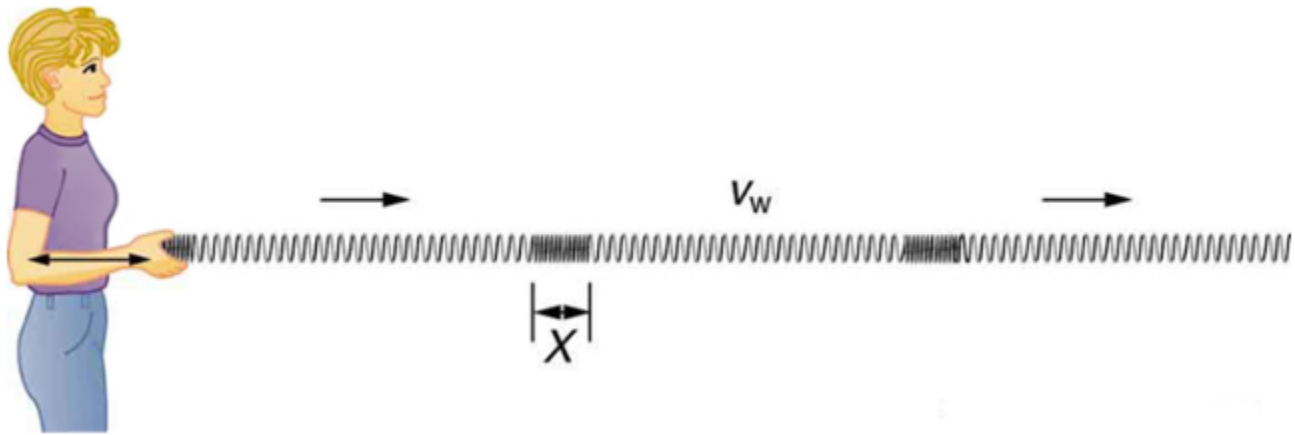


Figure 3: In this example of a longitudinal wave, the wave propagates horizontally, and the disturbance in the cord is also in the horizontal direction.

Waves may be transverse, longitudinal, or a combination of the two. (Water waves are actually a combination of transverse and longitudinal. The simplified water wave illustrated in Figure 1 shows no longitudinal motion of the bird.) The waves on the strings of musical instruments are transverse—so are electromagnetic waves, such as visible light.

Sound waves in air and water are longitudinal. Their disturbances are periodic variations in pressure that are transmitted in fluids. Fluids do not have appreciable shear strength, and thus the sound waves in them must be longitudinal or compressional. Sound in solids can be both longitudinal and transverse.

Earthquake waves under Earth's surface also have both longitudinal and transverse components (called compressional or P-waves and shear or S-waves, respectively). These components have important individual characteristics—they propagate at different speeds, for example. Earthquakes also have surface waves that are similar to surface waves on water.

BASIC PROPERTIES OF WAVES



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WAVE ON A STRING (SIMULATION)



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ENERGY IN WAVES: INTENSITY

All waves carry energy. The energy of some waves can be directly observed. Earthquakes can shake whole cities to the ground, performing the work of thousands of wrecking balls.

Loud sounds pulverize nerve cells in the inner ear, causing permanent hearing loss. Ultrasound is used for deep-heat treatment of muscle strains. A laser beam can burn away a malignancy. Water waves chew up beaches.

The amount of energy in a wave is related to its amplitude. Large-amplitude earthquakes produce large ground displacements. Loud sounds have higher pressure amplitudes and come from larger-amplitude source vibrations than soft sounds. Large ocean breakers churn up the shore more than small ones. More quantitatively, a wave is a displacement that is resisted by a restoring force.

The energy effects of a wave depend on time as well as amplitude. For example, the longer deep-heat ultrasound is applied, the more energy it transfers. Waves can also be concentrated or spread out. Sunlight, for example, can be focused to burn wood. Earthquakes spread out, so they do less damage the farther they get from the source. In both cases, changing the area the waves cover has important effects. All these pertinent factors are included in the definition of intensity I as power per unit area:

$$I = \frac{P}{A}$$

Where P is the power carried by the wave through area A . The definition of intensity is valid for any energy in transit, including that carried by waves. The SI unit for intensity is watts per square meter (W/m^2). For example, infrared and visible energy from the Sun impinge on Earth at an intensity of $1300 W/m^2$ just above the atmosphere.

There are other intensity-related units in use, too. The most common is the decibel. For example, a 90-decibel sound level corresponds to an intensity of 10^{-3} W/m^2 . (This quantity is not much power per unit area considering that 90 decibels is a relatively high sound level. Decibels will be discussed in some detail in a later chapter.)

DESCRIBE THE MOVEMENT OF SOUND WAVES IN THE ENVIRONMENT

In this section, you will learn how to understand the physics, speed, wavelength, intensity, and level of sound by reading each description along with watching the videos included. Also, short problems to check your understanding are included.

PHYSICS OF HEARING: SOUND

Sound can be used as a familiar illustration of waves. Because hearing is one of our most important senses, it is interesting to see how the physical properties of sound correspond to our perceptions of it. **Hearing** is the perception of sound, just as vision is the perception of visible light. The physical phenomenon of **sound** is defined to be a disturbance of matter that is transmitted from its source outward. Sound is a wave. The amplitude of a sound wave decreases with distance from its source, because the energy of the wave is spread over a larger and larger area. But it is also absorbed by objects, such as the eardrum.

A vibrating string produces a sound wave as illustrated in Figures 4, 5, and 6. As the string oscillates back and forth, it transfers energy to the air, mostly as thermal energy created by turbulence. But a small part of the string's energy goes into compressing and expanding the surrounding air, creating slightly higher and lower local pressures. These compressions (high pressure regions) and rarefactions (low pressure regions) move out as longitudinal pressure waves having the same frequency as the string—they are the disturbance that is a sound wave. (Sound waves in air and most fluids are longitudinal because fluids have almost no shear strength. In solids, sound waves can be both transverse and longitudinal.) Figure 6 shows a graph of gauge pressure versus distance from the vibrating string.

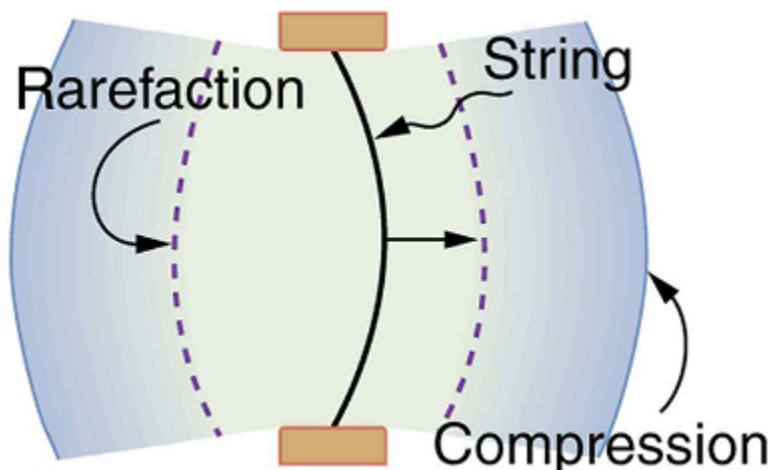


Figure 4: A vibrating string moving to the right compresses the air in front of it and expands the air behind it.

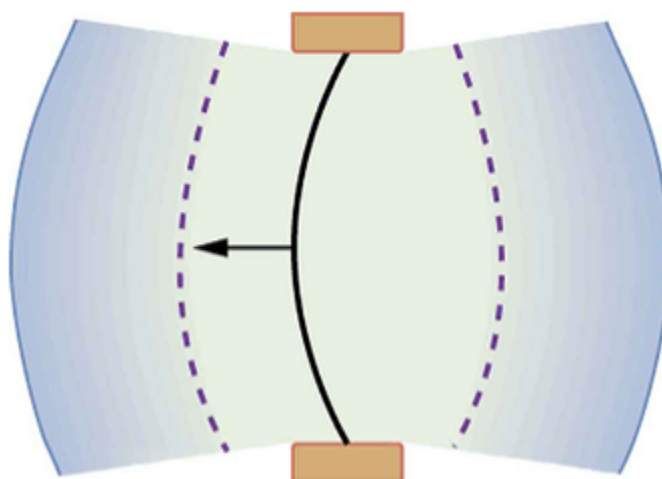


Figure 5: As the string moves to the left, it creates another compression and rarefaction as the ones on the right move away from the string.

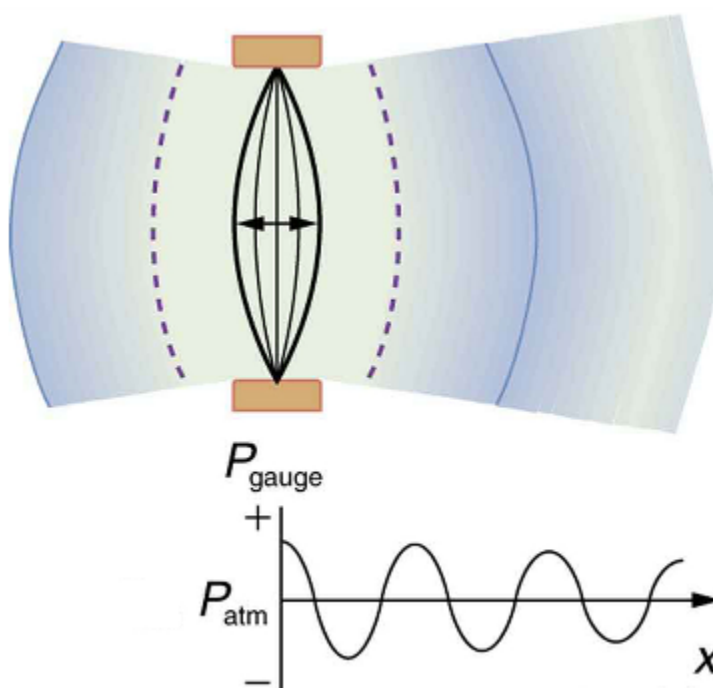


Figure 6: After many vibrations, these are a series of compressions and rarefactions moving out from the string as a sound wave. The graph shows gauge pressure versus distance from the source. Pressures vary only slightly from atmospheric for ordinary sounds.

SPEED OF SOUND, FREQUENCY, AND WAVELENGTH

Sound, like all waves, travels at a certain speed and has the properties of frequency and wavelength. You can observe direct evidence of the speed of sound while watching a fireworks display. The flash of an explosion is seen well before its sound is heard, implying both that sound travels at a finite speed and that it is much slower than light. You can also directly sense the frequency of a sound. Perception of frequency is called pitch. The wavelength of sound is not directly sensed, but indirect evidence

is found in the correlation of the size of musical instruments with their pitch. Small instruments, such as a piccolo, typically make high-pitch sounds, while large instruments, such as a tuba, typically make low-pitch sounds. High pitch means small wavelength, and the size of a musical instrument is directly related to the wavelengths of sound it produces. So, a small instrument creates short-wavelength sounds. Similar arguments hold that a large instrument creates long-wavelength sounds.

The relationship of the speed of sound, its frequency, and wavelength is the same as for all waves:

$$v_w = f\lambda$$

Where v_w is the speed of sound, f is its frequency, and λ is its wavelength. The wavelength of a sound is the distance between adjacent identical parts of a wave – for example, between adjacent compressions as illustrated in Figure 7. The frequency is the same as that of the source and is the number of waves that pass a point per unit of time.

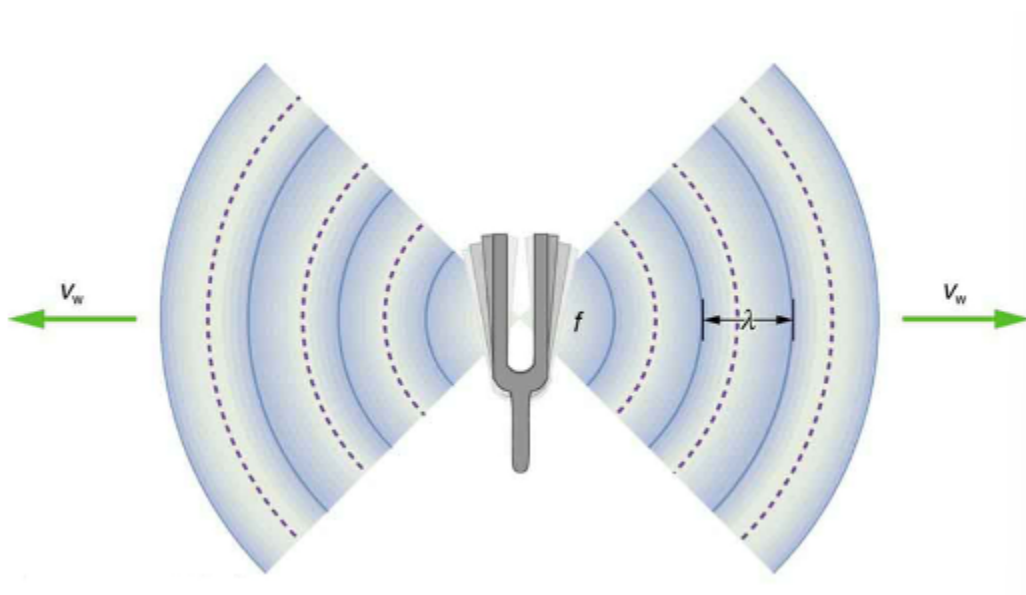


Figure 7: A sound wave emanates from a source vibrating at a frequency f , propagates at v_w , and has a wavelength λ

The speed of sound can change when sound travels from one medium to another. However, the frequency usually remains the same because it is like a driven oscillation and has the frequency of the original source. If v_w changes and f remains the same, then the wavelength λ must change. That is, because $v_w = f\lambda$, the higher the speed of a sound, the greater its wavelength for a given frequency.

Check Your Understanding: Speed of Sound, Frequency, and Wavelength



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<https://uta.pressbooks.pub/oert-mpsfundamentals/?p=85#h5p-7>

SOUND INTENSITY AND SOUND LEVEL

In a quiet forest, you can sometimes hear a single leaf fall to the ground. After settling into bed, you may hear your blood pulsing through your ears. But when a passing motorist has his stereo turned up, you cannot even hear what the person next to you in your car is saying. We are all very familiar with the loudness of sounds and aware that they are related to how energetically the source is vibrating. In cartoons depicting a screaming person (or an animal making a loud noise), the cartoonist often shows a gaping mouth with a vibrating uvula, the hanging tissue at the back of the mouth, to suggest a loud sound coming from the throat (see Figure 8 below). High noise exposure is hazardous to hearing, and it is common for musicians to have hearing losses that are sufficiently severe that they interfere with the musicians' abilities to perform. The relevant physical quantity is sound intensity, a concept that is valid for all sounds whether or not they are in the audible range.

Intensity is defined to be the power per unit area carried by a wave. Power is the rate at which energy is transferred by the wave. In equation form intensity I is $I = \frac{P}{A}$, where P is the power through an area A .

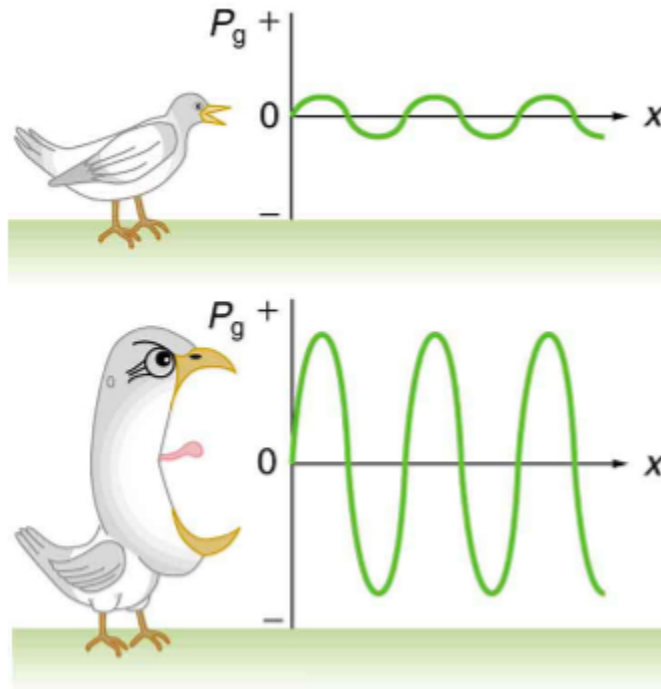


Figure 8: Graphs of the gauge pressures in two sound waves of different intensities. The more intense sound is produced by a source that has larger-amplitude oscillations and has greater pressure maxima and minima. Because pressures are higher in the greater-intensity sound, it can exert larger forces on the objects it encounters.

Sound intensity levels are quoted in decibels (dB) much more often than sound intensities in watts per meter squared. Decibels are the unit of choice in the scientific literature as well as in the popular media. The reasons for this choice of units are related to how we perceive sounds. How our ears perceive sound can be more accurately described by the logarithm of the intensity rather than directly to the intensity. The **sound intensity level** β in decibels of a sound having an intensity I in watts per meter squared is defined to be $\beta(\text{dB}) = 10 \log_{10} \left(\frac{I}{I_0} \right)$, where $I_0 = 10^{-12} \text{ W/m}^2$ is a reference intensity. In particular, I_0 is the lowest or threshold intensity of sound a person with normal hearing can perceive at a frequency of 1000 Hz. Sound intensity level is not the same as intensity. Because β is defined in terms of a ratio, it is a unitless quantity telling you the level of the sound relative to a fixed standard (10^{-12} W/m^2 , in this case). The units of decibels (dB) are used to indicate this ratio is multiplied by 10 in its definition. The bel, upon which the decibel is based, is named for Alexander Graham Bell, the inventor of the telephone.

Table 1: Sound Intensity Levels and Intensities ¹

Sound intensity level β (dB)	Intensity I (W/m ²)	Example/effect
00" > 0	1×10^{-12}	Threshold of hearing at 1000 Hz
1010" > 10	1×10^{-11}	Rustle of leaves
2020" > 20	1×10^{-10}	Whisper at 1 m distance
3030" > 30	1×10^{-9}	Quiet home
4040" > 40	1×10^{-8}	Average home
5050" > 50	1×10^{-7}	Average office, soft music
6060" > 60	1×10^{-6}	Normal conversation
7070" > 70	1×10^{-5}	Noisy office, busy traffic
8080" > 80	1×10^{-4}	Loud radio, classroom lecture
9090" > 90	1×10^{-3}	Inside a heavy truck; damage from prolonged exposure ¹
100100" > 100	1×10^{-2}	Noisy factory, siren at 30 m; damage from 8 h per day exposure
110110" > 110	1×10^{-1}	Damage from 30 min per day exposure
120120" > 120	11" > 1	Loud rock concert, pneumatic chipper at 2 m; threshold of pain
140140" > 140	1×10^2	Jet airplane at 30 m; severe pain, damage in seconds
160160" > 160	1×10^4	Bursting of eardrums

The decibel level of a sound having the threshold intensity of 10^{-12} W/m^2 is $\beta = 0 \text{ dB}$, because $\log_{10} 1 = 0$. That is, the threshold of hearing is 0 decibels. Table 1 gives levels in decibels and intensities in watts per meter squared for some familiar sounds.

Check Your Understanding: Sound Intensity and Sound Level



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1. Several government agencies and health-related professional associations recommend that 85 dB not be exceeded for 8-hour daily exposures in the absence of hearing protection.

WAVE INTERFERENCE (PHET SIMULATION)



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SOUND WAVES AND THEIR SOURCES (1933 MOVIE CLIP)



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ACOUSTICS AND YOUR ENVIRONMENT: THE BASICS OF SOUNDS AND HIGHWAY TRAFFIC NOISE (48 MIN)



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DESCRIBE THE DOPPLER EFFECT IN THE CONTEXT OF WAVES EMANATING FROM A MOVING SOURCE

The characteristic sound of a motorcycle buzzing by is an example of the Doppler effect. The high-pitch scream shifts dramatically to a lower-pitch roar as the motorcycle passes by a stationary observer. The closer the motorcycle brushes by, the more abrupt the shift. The faster the motorcycle moves, the greater the shift. We also hear this characteristic shift in frequency for passing race cars, airplanes, and trains.

The Doppler effect is an alteration in the observed frequency of a sound due to motion of either the source or the observer. Although less familiar, this effect is easily noticed for a stationary source and moving observer. For example, if you ride a train past a stationary warning bell, you will hear the bell's frequency shift from high to low as you pass by. The actual change in frequency due to relative motion of source and observer is called a Doppler shift. The Doppler effect and Doppler shift are named for the Austrian physicist and mathematician Christian Johann Doppler (1803–1853), who did experiments with both moving sources and moving observers. Doppler, for example, had musicians

play on a moving open train car and also play standing next to the train tracks as a train passed by. Their music was observed both on and off the train, and changes in frequency were measured.

What causes the Doppler shift? Figures 9, 10, and 11 compare sound waves emitted by stationary and moving sources in a stationary air mass. Each disturbance spreads out spherically from the point where the sound was emitted. If the source is stationary, then all of the spheres representing the air compressions in the sound wave centered on the same point, and the stationary observers on either side see the same wavelength and frequency as emitted by the source, as seen in Figure 9. If the source is moving, as in Figure 10, then the situation is different. Each compression of the air moves out in a sphere from the point where it was emitted, but the point of emission moves. This moving emission point causes the air compressions to be closer together on one side and farther apart on the other. Thus, the wavelength is shorter in the direction the source is moving (on the right in Figure 10), and longer in the opposite direction (on the left in Figure 10). Finally, if the observers move, as in Figure 11, the frequency at which they receive the compressions changes. The observer moving toward the source receives them at a higher frequency, and the person moving away from the source receives them at a lower frequency.

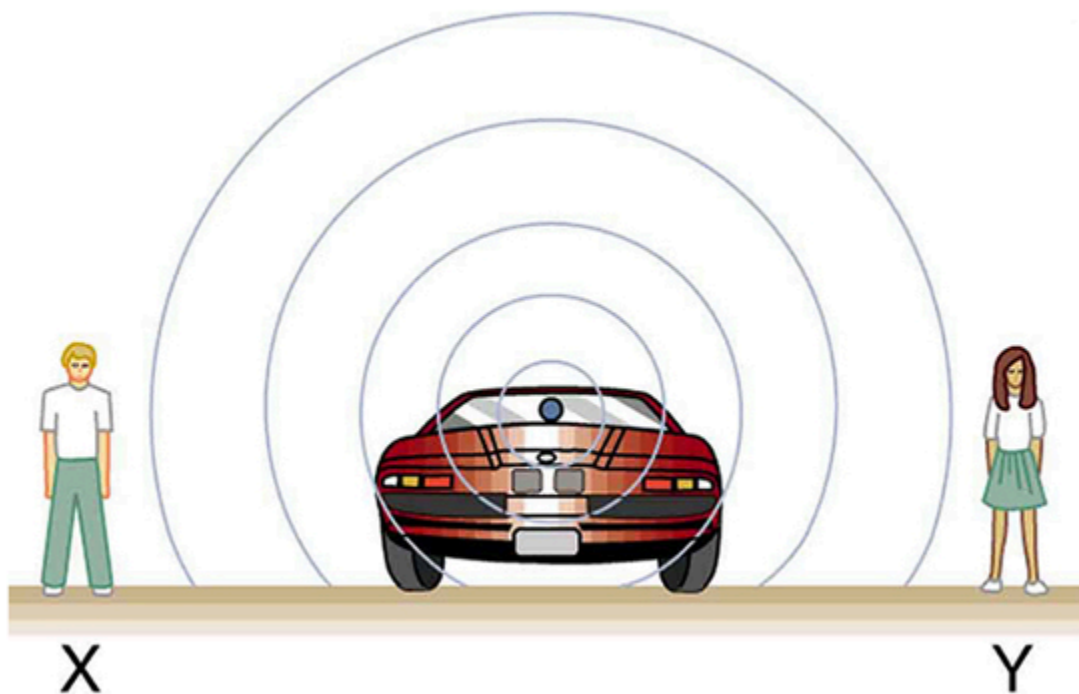


Figure 9: Sounds emitted by a source spread out in spherical waves. Because the source, observers, and air are stationary, the wavelength and frequency are the same in all directions and to all observers.

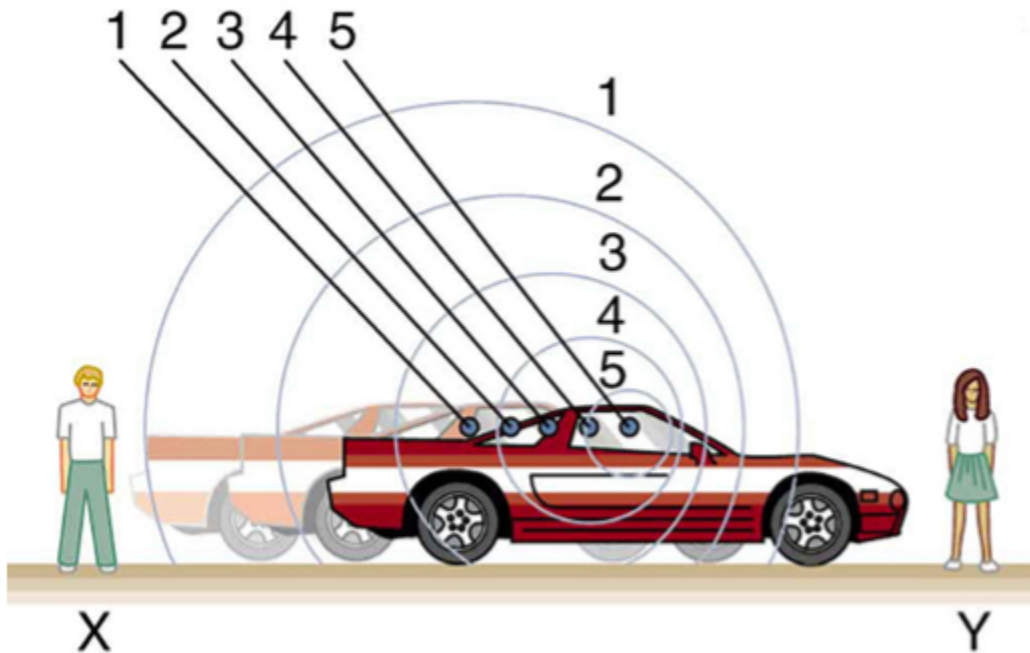


Figure 10: Sounds emitted by a source moving to the right spread out from the points at which they were emitted. The wavelength is reduced and, consequently, the frequency is increased in the direction of motion, so that the observer on the right hears a higher-pitch sound. The opposite is true for the observer on the left, where the wavelength is increased, and the frequency is reduced.

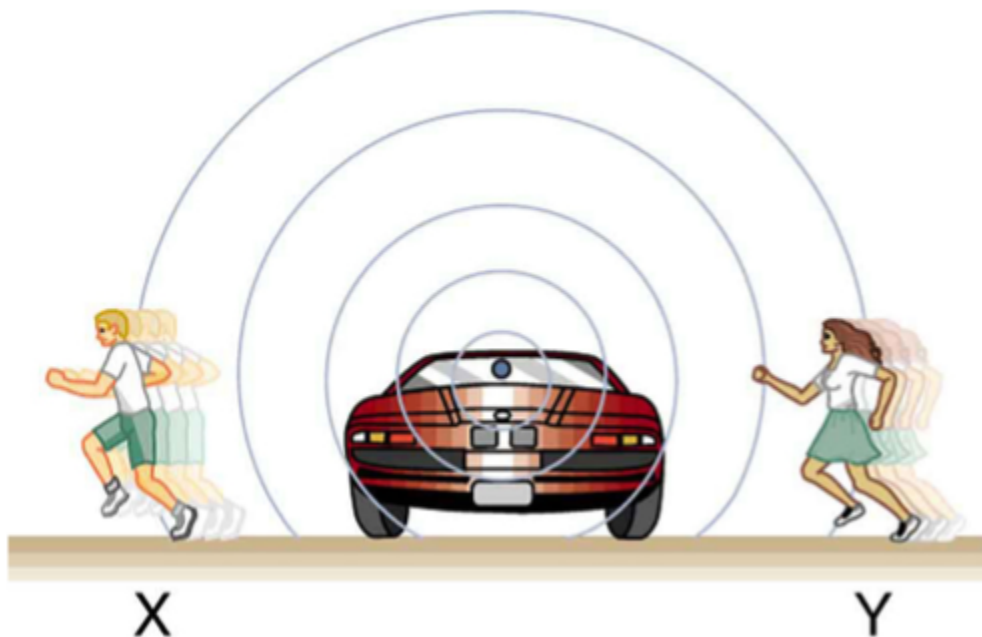


Figure 11: The same effect is produced when the observers move relative to the source. Motion toward the source increases frequency as the observer on the right passes through more wave crests than she would if stationary. Motion away from the source decreases frequency as the observer on the left passes through fewer wave crests, he would if stationary.

We know that wavelength and frequency are related by $v_w = f\lambda$, where v_w is the fixed speed of sound. The sound moves in a medium and has the same speed v_w in that medium whether the source

is moving or not. The f multiplied by λ is a constant. Because the observer on the right in the Figure 10 receives a shorter wavelength, the frequency she receives must be higher. Similarly, the observer on the left receives a longer wavelength, and hence he hears a lower frequency. The same thing happens in Figure 11. A higher frequency is received by the observer moving toward the source, and a lower frequency is received by an observer moving away from the source. In general, then, relative motion of source and observer toward one another increases the received frequency. Relative motion apart decreases frequency. The greater the relative speed is, the greater the effect.

WHAT IS THE DOPPLER EFFECT?



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Check Your Understanding: What is the Doppler Effect?



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IDENTIFY THE CAUSES OF NOISE FROM THE TRANSPORTATION SYSTEM

In this section, you will learn about the causes of transportation system noise. To gain a better understanding of the many different noises associated with our environment, please read the article “Traffic Noise and Transportation” by The Center for Environmental Excellence by the American Association of State Highway and Transportation Officials (AASHTO).

ADSORPTIVE SOUND WALL SYSTEMS



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RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

In this section, you will learn about spot speed study and speed limit enforcement helping ensure that roadway design is consistent with speed on roadways and highways. You will also learn about sustainable transportation planning and development.

SPOT SPEED STUDY AND SPEED LIMIT ENFORCEMENT

The information from spot speed studies is critical for the enforcement and implementation of speed management strategies if needed. A Doppler radar, either hand-held, vehicle-mounted, or poll-mounted, has been one of the most commonly used tools for speed limit enforcement. These devices are also useful in conducting spot speed studies to ensure that the roadway design is consistent with the desired speed for the street and highways. Newer Lidar-based speed measurement also works based on the Doppler Effect discussed in this chapter.

TRANSPORTATION PLANNING AND SUSTAINABLE DEVELOPMENT

As we learn about the ill effects of noise pollution, it becomes a vital part of the environmental impact assessments and mitigation for transportation projects. It remains a critical equity concern since, historically, lower-income and minoritized communities bear the brunt of the impact of traffic-related noise. Traffic noise barriers are one of the key mitigation measures that manipulate sound waves to reduce their intensity and help minimize their impact. Noise barriers reduce the sound entering a community by either absorbing/reflecting the sound of traffic or forcing the waves to take an alternate path over and around the obstacle. The design of noise barriers is based on the wave propagation principles outlined in this chapter.

Key Takeaways

- The technology used for spot speed studies and speed limit enforcement is based on the Doppler effect in wave propagation. Spot speed studies are critical to ensure that the roadway design context aligns with the speed limit posted on the roadway segment.
- The design of noise barriers to protect communities surrounding freeways and railroads from noise pollution is based on the wave propagation principles discussed in this chapter. It has been an equity issue since marginalized lower-income communities have historically borne the most impact of roadway noise pollution.

GLOSSARY

Amplitude: the maximum displacement from the equilibrium position of an object oscillating around the equilibrium position

Doppler effect: an alteration in the observed frequency of a sound due to motion of either the source or the observer

Doppler shift: the actual change in frequency due to relative motion of source and observer

Frequency: number of events per unit of time

Hearing: the perception of sound

Intensity: power per unit area; sounds below 20 Hz

Longitudinal wave: a wave in which the disturbance is parallel to the direction of propagation

Loudness: the perception of sound intensity

Natural frequency: the frequency at which a system would oscillate if there were no driving and no damping forces

Period: time it takes to complete one oscillation

Pitch: the perception of the frequency of a sound

Sound: a disturbance of matter that is transmitted from its source outward

Sound intensity level: a unitless quantity telling you the level of the sound relative to a fixed standard

Sound pressure level: the ratio of the pressure amplitude to a reference pressure

Tone: number and relative intensity of multiple sound frequencies

Transverse wave: a wave in which the disturbance is perpendicular to the direction of propagation

Wave: a disturbance that moves from its source and carries energy

Wave velocity: the speed at which the disturbance moves. Also called the propagation velocity or propagation speed

Wavelength: the distance between adjacent identical parts of a wave

MEDIA ATTRIBUTIONS

NOTE: Text by Greg Wolfe, et al.: Access for free at <https://openstax.org/books/college-physics-ap-courses/pages/1-connection-for-ap-r-courses>

VIDEOS

- Video 1: “Basic Properties of Waves” by Animations for Physics and Astronomy is licensed under Creative Commons Attribution-ShareAlike 3.0 Unported (CC BY-SA 3.0)
- Video 2: “Sound Waves and Their Sources (1933) Remastered” by Sterocanvas is licensed under Creative Commons Attribution-ShareAlike 3.0 Unported (CC BY-SA 3.0)
- Video 3: “Acoustics and Your Environment: The Basis of Sound and Highway Traffic Noise” by PublicResourceOrg is licensed under Creative Commons Attribution-ShareAlike 3.0 Unported (CC BY-SA 3.0)
- Video 4: “What is the Doppler Effect?” by Engineering Technology Simulation Learning Videos is licensed under Creative Commons Attribution-ShareAlike 3.0 Unported (CC BY-SA 3.0)
- Video 5: “Adsorptive Sound Wall Systems” by SoundFighterWalls is licensed under Creative Commons Attribution-ShareAlike 3.0 Unported (CC BY-SA 3.0)

FIGURES

- Figure 1: “Chapter 16: Oscillatory Motion and Waves” by Greg Wolfe, Erika Gasper, John

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- Figure 11: “Chapter 17: Physics of Hearing” by Greg Wolfe, Erika Gasper, John Stoke, Julie Kretchman, David Anderson, Nathan Czuba, Sudhi Oberoi, Liza Pujji, Irina Lyublinskaya, and Douglas Ingram is licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)

TABLES

- Table 1: Garnett, A. (2022, April 26). Traffic noise overview.Center for Environmental Excellence | AASHTO. Retrieved July 1, 2022, from <https://environment.transportation.org/education/environmental-topics/traffic-noise/traffic-noise-overview/>

SIMULATIONS

- Simulation 1: “Wave on a String” by PhET Simulations is licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)
- Simulation 2: “Wave Interference” by PhET Simulations is licensed under Creative Commons Attribution 4.0 International (CC BY 4.0)

REFERENCES

“Traffic Noise and Transportation” by The Center for Environmental Excellence by the American Association of State Highway and Transportation Officials (AASHTO).

CHAPTER 8: PROBABILITY: BASIC PRINCIPLES AND DISTRIBUTIONS

The transportation system operations and planning are critically dependent on understanding the basic principles of probability. Several processes are modeled using probability distributions for real-valued random variables. These distributions include normal distribution for the speed of vehicles on the road, Poisson's distribution for gaps in traffic on an uncongested facility, or negative binomial as a distribution for crash frequency on a roadway segment.

Learning Objectives

- Use basic counting techniques (multiplication rule, combinations, permutations) to estimate probability and odds.
- Set up and work with distributions for discrete random variables, including Bernoulli, binomial, geometric, and Poisson distributions.
- Set up and work with distributions for continuous random variables, including uniform, normal and exponential distributions.
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter

USE BASIC COUNTING TECHNIQUES TO ESTIMATE PROBABILITY AND ODDS

This section will explain ways to estimate probability and odds with videos to help your understanding. Also, short problems to check your understanding are included.

MULTIPLICATION RULE

Multiplication Rule

The multiplication rule says: If there are n ways to perform action 1 and then by m ways to perform action 2, then there are $n \cdot m$ ways to perform action 1 followed by action 2.

Count Outcomes Using Tree Diagram



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Check Your Understanding: Multiplication Rule



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PERMUTATIONS

Permutations

A permutation of a set is a particular ordering of its elements. For example, the set $\{a, b, c\}$ has six permutations: abc , acb , bac , bca , cab , cba . We found the number of permutations by listing them all. We could have also found the number of permutations by using the multiplication rule. That is, there are 3 ways to pick the first element, then 2 ways for the second, and 1 for the third. This gives a total of $3 \cdot 2 \cdot 1 = 6$ permutations.

In general, the multiplication rule tells us that the number of permutations of a set of elements is

$$k! = k \cdot (k - 1) \cdots 3 \cdot 2 \cdot 1$$

We also talk about the permutations of k things out of a set of n things.

Example: List all the permutations of 3 elements out of the set $\{a, b, c, d\}$

Table 1

<i>abc</i>	<i>acb</i>	<i>bac</i>	<i>bca</i>	<i>cab</i>	<i>cba</i>
<i>abd</i>	<i>adb</i>	<i>bad</i>	<i>bda</i>	<i>dab</i>	<i>dba</i>
<i>acd</i>	<i>adc</i>	<i>cad</i>	<i>cda</i>	<i>dac</i>	<i>dca</i>
<i>bcd</i>	<i>bdc</i>	<i>cbd</i>	<i>cdb</i>	<i>dbc</i>	<i>dcb</i>

Note that abc and acb count as distinct permutations. That is, for permutations the order matters.

There are 24 permutations. Note that the multiplication rule would have told us there are $4 \cdot 3 \cdot 2 = 24$ permutations without bothering to list them all.

Permutation Formula



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Zero Factorial



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Ways to Pick Officers - Example



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Check Your Understanding: Permutations



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COMBINATIONS

Combinations

In contrast to permutations, in combinations order does not matter: permutations are lists and combinations are sets.

Example: List all the combinations of 3 elements out of the set $\{a, b, c, d\}$.

Answer: Such a combination is a collection of 3 elements without regard to order. So, abc and cab both represent the same combination. We can list all the combinations by listing all the subsets of exactly 3 elements.

$\{a, b, c\}$ $\{a, b, d\}$ $\{a, c, d\}$ $\{b, c, d\}$

There are only 4 combinations. Contrast this with the 24 permutations in the previous example. The factor of 6 comes because every combination of 3 things can be written in 6 different orders.

Introduction to Combinations



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Combination Formula



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Combination Example: 9 Card Hands



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Check Your Understanding: Combinations



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PERMUTATIONS AND COMBINATIONS COMPARISON

We will use the following notations.

${}_nP_k$ = number of permutations (list) of k distinct elements from a set of size n

${}_nC_k = \binom{n}{k}$ = number of combinations (subsets) of k elements from a set of size n

We emphasize that by the number of combinations of k elements we mean the number of subsets of size k .

These have the following notation and formulas:

Permutations: ${}_nP_k = \frac{n!}{(n-k)!} = n(n-1) \dots (n-k+1)$

Combinations: ${}_nC_k = \frac{n!}{k!(n-k)!} = \frac{{}_nP_k}{k!}$

The notation ${}_nC_k$ is read “ n choose k ”. The formula for ${}_nP_k$ follows from the multiplication rule. It also implies the formula for ${}_nC_k$ because a subset of size k can be ordered in $k!$ ways.

We can illustrate the relationship between permutations and combinations by lining up the results of the previous two examples.

Permutations: ${}_4P_3$

Table 2

<i>abc</i>	<i>acb</i>	<i>bac</i>	<i>bca</i>	<i>cab</i>	<i>cba</i>
<i>abd</i>	<i>adb</i>	<i>bad</i>	<i>bda</i>	<i>dab</i>	<i>dba</i>
<i>acd</i>	<i>adc</i>	<i>cad</i>	<i>cda</i>	<i>dac</i>	<i>dca</i>
<i>bcd</i>	<i>bdc</i>	<i>cbd</i>	<i>cdb</i>	<i>bdc</i>	<i>dcb</i>

Combinations: ${}_4C_3$

$\{a, b, c\}$
 $\{a, b, d\}$
 $\{a, c, d\}$
 $\{b, c, d\}$

Notice that each row in the permutations list consists of all $3!$ permutations of the corresponding set in the combinations list.

Check Your Understanding: Permutations and Combinations Comparison



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PROBABILITY USING THE RULES

The General Multiplication Rule

When we calculate probabilities involving one event AND another event occurring, we multiply their probabilities.

In some cases, the first event happening impacts the probability of the second event. We call these **dependent events**.

In other cases, the first event happening does not impact the probability of the seconds. We call these **independent events**.

Independent events: Flipping a coin twice

What is the probability of flipping a fair coin and getting “heads” twice in a row? That is, what is the probability of getting heads on the first flip AND heads on the second flip?

Imagine we had 100 people simulate this and flip a coin twice. On average, 50 people would get heads on the first flip, and then 25 of them would get heads again. So, 25 out of the original 100 people – or $1/4$ of them – would get heads twice in a row.

The number of people we start with does not really matter. Theoretically, $1/2$ of the original group will get heads, and $1/2$ of that group will get heads again. To find a fraction of a fraction, we multiply.

We can represent this concept with a tree diagram like the one shown below in Figure 1.

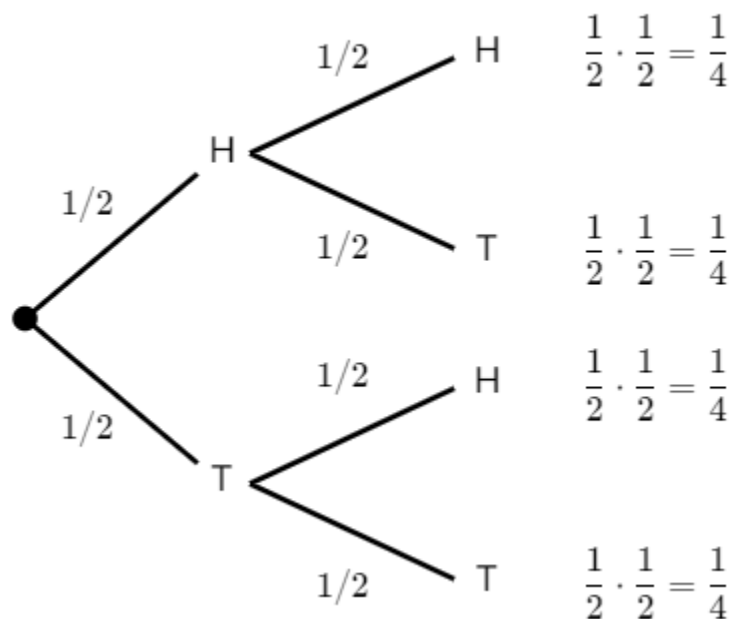


Figure 1

We multiply the probabilities along the branches to find the overall probability of one event AND the next event occurring.

For example, the probability of getting two “tails” in a row would be:

$$P(T \text{ and } T) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

When two events are **independent**, we can say that

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Be careful! This formula only applies to independent events.

Dependent events: Drawing cards

We can use a similar strategy even when we are dealing with dependent events.

Consider drawing two cards, without replacement, from a standard deck of 52 cards. That means we are drawing the first card, leaving it out, and then drawing the second card.

What is the probability that both cards selected are black?

Half of the 52 cards are black, so the probability that the first card is black is $26/52$. But the probability of getting a black card changes on the next draw, since the number of black cards and the total number of cards have both been decreased by 1.

Here is what the probabilities would look like in a tree diagram:

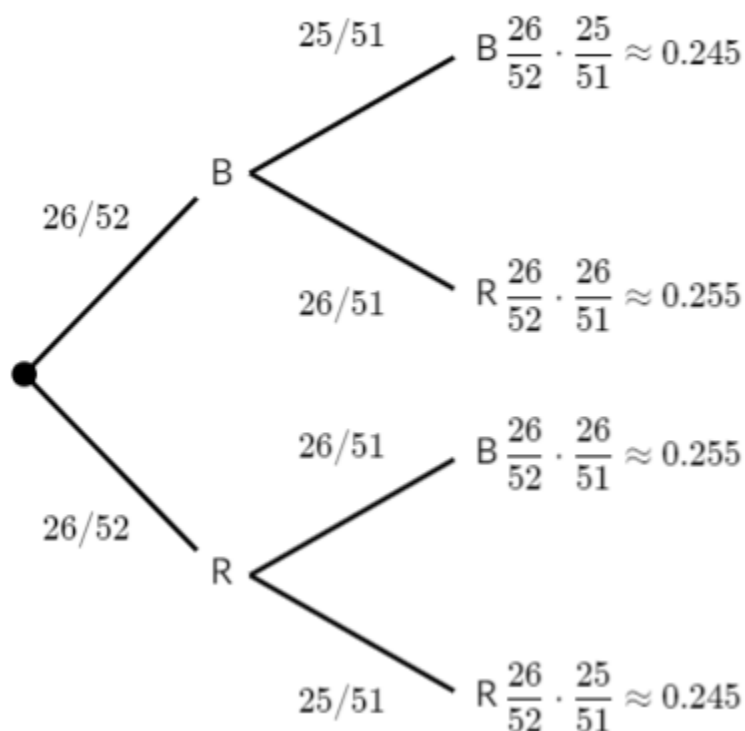


Figure 2

So, the probability that both cards are black is:

$$P(\text{both black}) = \frac{26}{52} \cdot \frac{25}{51} \approx 0.245$$

The General Multiplication Rule

For any two events, we can say that

$$P(A \text{ and } B) = P(A) \cdot P(B | A)$$

The vertical bar in $P(B | A)$ means “given,” so this could also be read as “the probability that B occurs *given* that A has occurred.”

This formula says that we can multiply the probabilities of two events, but we need to take the first event into account when considering the probability of the second event.

If the events are independent, one happening does not impact the probability of the other, and in that case, $P(B | A) = P(B)$.

Check Your Understanding: Probability Using the Rules



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Probability Using Combinations



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Probability and Combinations



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Conditional Probability and Bayes' Theorem

Conditional Probability

Conditional probability answers the questions 'how does the probability of an event change if we have extra information.'

Example: Toss a fair coin 3 times.

What is the probability of 3 heads?

Answer: Sample space = $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. All outcomes are equally likely, so $P(3 \text{ heads}) = \frac{1}{8}$

Suppose we are told that the first toss was heads. Given this information how should we compute the probability of 3 heads?

Answer: We have a new (reduced) sample space = $\{HHH, HHT, HTH, HTT\}$. All outcomes are equally likely, so $P(3 \text{ heads given that the first toss is heads}) = \frac{1}{4}$

This is called **conditional probability**, since it takes into account additional conditions. To develop the notation, we rephrase (b) in terms of events.

Rephrased (b): Let A be the event ‘all three tosses are heads’ = $\{HHH\}$. Let B be the event ‘the first toss is heads’ = $\{HHH, HHT, HTH, HTT\}$.

The conditional probability of A knowing that B occurred is written

$$P(A \mid B)$$

This is read as ‘the conditional probability of A given B ’ **OR** ‘the probability of A conditioned on B ’ **OR simply** ‘the probability of A given B .’

We can visualize conditional probability as follows. Think of $P(A)$ as the proportion of the area of the *whole* sample space taken up by A . For $P(A \mid B)$ we restrict our attention to B . That is, $P(A \mid B)$ is the proportion of area of B taken up by A , i.e., $P(A \cap B)/P(B)$.

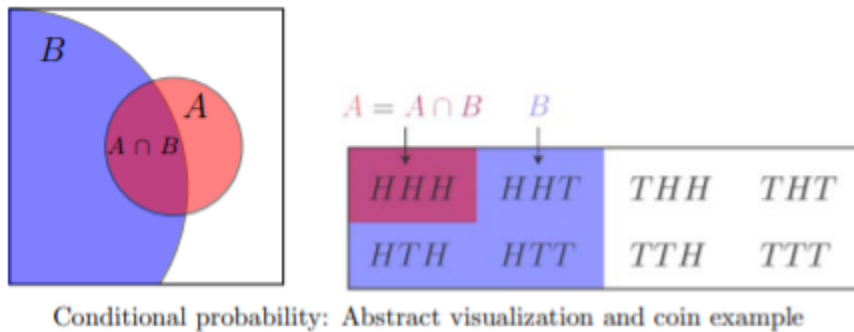


Figure 3

The formal definition of conditional probability catches the gist of the above example and visualization.

Formal definition of conditional probability

Let A and B be events. We define the conditional probability of A given B as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0$$

Let us redo the coin-tossing example using the definition in the equation above. Recall $A = 3 \text{ heads}$ and $B = \text{first toss is heads}$. We have $P(A) = 1/8$ and $P(B) = 1/2$. Since $A \cap B = A$, we also have $P(A \cap B) = 1/8$. Now according to the equation, $P(A \mid B) = \frac{1/8}{1/2} = \frac{1}{4}$ which agrees with the answer in Example B.

Bayes' Theorem

Bayes' theorem is a pillar of both probability and statistics. For two events A and B Bayes' theorem says

$$P(B | A) = \frac{P(A|B) \cdot P(B)}{P(A)}$$

Bayes' rule tells us how to 'invert' conditional probabilities, i.e., to find $P(B | A)$ from $P(A | B)$.

Proof of Bayes' Theorem

The key point is that $A \cap B$ is symmetric in A and B . So, the multiplication rule says $P(B | A) \cdot P(A) = P(A \cap B) = P(A | B) \cdot P(B)$.

Now divide through by $P(A)$ to get Bayes' rule.

A common mistake is to confuse $P(A | B)$ and $P(B | A)$. They can be very different. This is illustrated in the next example.

Example: Toss a coin 5 times. Let H_1 = first toss is heads and let H_A = all 5 tosses are heads. Then $P(H_1 | H_A) = 1$ but $P(H_A | H_1) = 1/16$.

For practice, let us use Bayes' theorem to compute $P(H_1 | H_A)$ using $P(H_A | H_1)$. The terms are $P(H_A | H_1) = 1/16$, $P(H_1) = 1/2$, $P(H_A) = 1/32$. So,

$$P(H_1 | H_A) = \frac{P(H_A|H_1)P(H_1)}{P(H_A)} = \frac{(1/16) \cdot (1/2)}{1/32} = 1$$

Which agrees with our previous calculation.

Conditional Probability and Combinations



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Check Your Understanding: Conditional Probability and Bayes' Theorem



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SET UP AND WORK WITH DISTRIBUTIONS FOR DISCRETE RANDOM VARIABLES

The following sections will help you become familiar with distributions for discrete and continuous random variables. The videos help explain random, discrete, and continuous variables. Probabilities are also explained through watching the videos in this section. Also, short problems to check your understanding are included.

RANDOM VARIABLES



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Discrete and Continuous Random Variables



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INTRODUCTION TO DISCRETE RANDOM VARIABLES

A student takes a ten-question, true-false quiz. Because the student had such a busy schedule, he or she could not study and guesses randomly at each answer. What is the probability of the student passing the test with at least a 70?

Small companies might be interested in the number of long-distance phone calls their employees make during the peak time of the day. Suppose the historical average is 20 calls. What is the probability that the employees make more than 20 long-distance phone calls during the peak time?

These two examples illustrate two different types of probability problems involving discrete random variables. Recall that discrete data are data that you can count, that is, the random variable can only take on whole number values. A **random variable** describes the outcomes of a statistical experiment in words. The values of a random variable can vary with each repetition of an experiment, often called a trial.

Random Variable Notation

The upper-case letter X denotes a random variable. Lowercase letters like x or y denote the value of a random variable. If X is a random variable, then X is written in words, and x is given as a number.

For example, let X = the number of heads you get when you toss three fair coins. The sample space for the toss of three fair coins is TTT; THH; HTH; HHT; HTT; THT; TTH; HHH. Then, $x = 0, 1, 2, 3$. X is in words and x is a number. Notice that for this example, the x values are countable outcomes. Because you can count the possible values as whole numbers that X can take on and the outcomes are random (the x values 0, 1, 2, 3), X is a discrete random variable.

Probability Density Functions (PDF) for a Random Variable

A **probability density function** or **probability distribution function** has two characteristics:

1. Each probability is between zero and one, inclusive.
2. The sum of the probabilities is one.

A probability density function is a mathematical formula that calculates probabilities for specific types of events. There is a sort of magic to a probability density function (Pdf) partially because the same formula often describes very different types of events. For example, the binomial Pdf will calculate probabilities for flipping coins, yes/no questions on an exam, opinions of voters in an up or down opinion poll, and indeed any binary event. Other probability density functions will provide probabilities for the time until a part will fail, when a customer will arrive at the turnpike booth, the number of telephone calls arriving at a central switchboard, the growth rate of a bacterium, and on and on. There are whole families of probability density functions that are used in a wide variety of applications, including medicine, business and finance, physics, and engineering, among others.

Counting Formulas and the Combinational Formula

To repeat, the probability of event A , $P(A)$, is simply the number of ways the experiment will result in A , relative to the total number of possible outcomes of the experiment.

As an equation this is:

$$P(A) = \frac{\text{number of ways to get } A}{\text{Total number of possible outcomes}}$$

When we looked at the sample space for flipping 3 coins, we could easily write the full sample space and thus could easily count the number of events that met our desired result, e.g., $x = 1$, where X is the random variable defined as the number of heads.

As we have larger numbers of items in the sample space, such as a full deck of 52 cards, the ability to write out the sample space becomes impossible.

We see that probabilities are nothing more than counting the events in each group we are interested in and dividing by the number of elements in the universe, or sample space. This is easy enough if we are counting sophomores in a Stat class, but in more complicated cases listing all the possible outcomes may take a lifetime. There are, for example, 36 possible outcomes from throwing just two six-sided dice where the random variable is the sum of the number of spots on the up-facing sides. If there were four dice, then the total number of possible outcomes would become 1,296. There are more than 2.5 MILLION possible 5-card poker hands in a

standard deck of 52 cards. Obviously keeping track of all these possibilities and counting them to get at a single probability would be tedious at best.

An alternative to listing the complete sample space and counting the number of elements we are interested in is to skip the step of listing the sample space, and simply figure out the number of elements in it and do the appropriate division. If we are after a probability, we really do not need to see each and every element in the sample space, we only need to know how many elements are there. Counting formulas were invented to do just this. They tell us the number of unordered subsets of a certain size that can be created from a set of unique elements. By unordered it is meant that, for example, when dealing cards, it does not matter if you got {ace, ace, ace, ace, king} or {king, ace, ace, ace, ace} or {ace, king, ace, ace, ace} and so on. Each of these subsets are the same because they each have 4 aces and one king.

Combinational Formula (Review)

$$\binom{n}{x} = {}_nC_x = \frac{n!}{x!(n-x)!}$$

It is also sometimes referred to as the Binomial Coefficient.

Let us find the hard way the total number of combinations of the four aces in a deck of cards if we were going to take them two at a time. The sample space would be:

S={ (Spade, Heart), (Spade, Diamond), (Spade, Club), (Diamond, Club), (Heart, Diamond), (Heart, Club) }

There are 6 combinations; formally, six unique unordered subsets of size 2 that can be created from 4 unique elements. To use the combinatorial formula, we would solve the formula as follows:

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6$$

If we wanted to know the number of unique 5-card poker hands that could be created from a 52-card deck, we simply compute:

$$\binom{52}{5}$$

where 52 is the total number of unique elements from which we are drawing and 5 is the size group we are putting them into.

With the combinatorial formula we can count the number of elements in a sample space without having to write each one of them down, truly a lifetime's work for just the number of 5 card hands from a deck of 52 cards.

Remember, a probability density function computes probability for us. We simply put the

appropriate numbers in the formula, and we get the probability of specific events. However, for these formulas to work they must be applied only to cases for which they were designed.

Constructing a Probability Distribution for Random Variable



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Probability with Discrete Random Variable Example



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Check Your Understanding: Random Variables



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Mean (expected value) of a Discrete Random Variable



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Expected Value

Example: Suppose we have a six-sided die marked with five 3's and one 6. What would you expect the average of 6000 rolls to be?

Answer: If we knew the value of each roll, we could compute the average by

summing the 6000 values and dividing by 6000. Without knowing the values, we can compute the **expected average** as follows.

Since there are five 3's and one 6, we expect roughly 5/6 of the rolls will give 3 and 1/6 will give 6. Assuming this to be exactly true, we have the following table of values and counts (Table 3):

Table 3			
Value:	3	6	
Expected counts:	5000	1000	

The average of these 6000 values is then

$$\frac{5000 \cdot 3 + 1000 \cdot 6}{6000} = \frac{5}{6} \cdot 3 + \frac{1}{6} \cdot 6 = 3.5$$

We consider this the expected average in the sense that we “expect” each of the possible values to occur with the given frequencies.

Definition: Suppose X is a discrete random variable that takes values x_1, x_2, \dots, x_n with probabilities $p(x_1), p(x_2), \dots, p(x_n)$. The **expected value** of X is denoted $E(X)$ and defined by

$$E(X) = \sum_{j=1}^n p(x_j) x_j = p(x_1) x_1 + p(x_2) x_2 + \dots + p(x_n) x_n$$

Notes:

The expected value is also called the mean or average of X and is often denoted by μ (“mu”).

As seen in the above example, the expected value need not be a possible value of the random variable. Rather it is a weighted average of the possible values.

Expected value is a summary statistic, providing a measure of the location or central tendency of a random variable.

If all the values are equally probable, then the expected value is just the usual average of the values.

Probability Mass Function and Cumulative Distribution Function

It gets tiring and hard to read and write $P(X = a)$ for the probability that $X = a$. When we know we are talking about X we will simply write $p(a)$. If we want to make X explicit we will write $p_X(a)$.

Definition: The probability mass function (pmf) of a discrete random variable is the function $p(a) = P(X = a)$.

Note:

1. We always have $0 \leq p(a) \leq 1$.
2. We allow a to be any number. If a is a value that X never takes, then $p(a) = 0$.

Mean and Center of Mass

You may have wondered why we use the name “probability mass function.” Here is the reason: if we place an object of mass $p(x_j)$ at position $x_1 = 3$ for each j , then $E(X)$ is the position of the center of mass. Let us recall the latter notion via an example.

Example: Suppose we have two masses along the x -axis, mass $m_1 = 500$ at position $x_1 = 3$ and mass $m_2 = 100$ at position $x_2 = 6$. Where is the center of mass?

Answer: Intuitively we know that the center of mass is closer to the larger mass.



Figure 4

From physics, we know the center of mass is

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{500 \cdot 3 + 100 \cdot 6}{600} = 3.5$$

We call this formula a “weighted” average of the x_1 and x_2 . Here x_1 is weighted more heavily because it has more mass.

Now look at the definition of expected value $E(X)$. It is a weighted average of the values of X with the weights being probabilities $p(x_i)$ rather than masses! We might say that “The expected value is the point at which the distribution would balance.” Note the similarity between the physics example and the previous dice example.

Check Your Understanding: Mean of a Discrete Random Variable



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Variance and Standard Deviation for Discrete Random Variables

The expected value (mean) of a random variable is a measure of location or central tendency. If you had to summarize a random variable with a single number, the mean would be a desirable choice. Still, the mean leaves out a good deal of information. For example, the random variables X and Y below both have mean 0, but their probability mass is spread out about the mean quite differently.

values X	-2	-1	0	1	2	values Y	-3	3
pmf $p(x)$	1/10	2/10	4/10	2/10	1/10	pmf $p(y)$	1/2	1/2

It is probably a little easier to see the different spreads in plots of the probability mass functions. We use bars instead of dots to give a better sense of the mass.

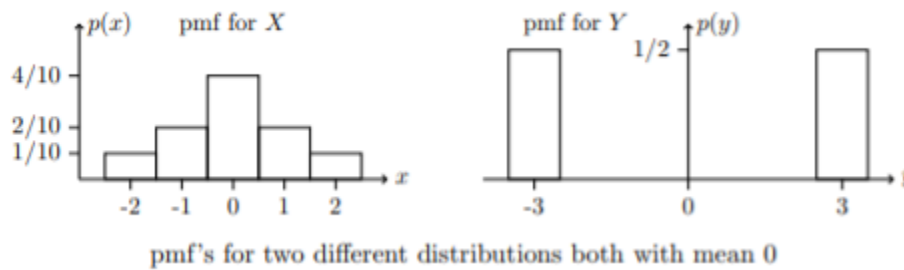


Figure 5

Variance and Standard Deviation

Taking the mean as the center of a random variable's probability distribution, the **variance** is a measure of how much the probability mass is **spread** out around this center.

Definition: If X is a random variable with mean $E(X) = \mu$, then the variance of X is defined by

$$\text{Var}(X) = E((X - \mu)^2)$$

The **standard deviation** σ of X is defined by

$$\sigma = \sqrt{\text{Var}(X)}$$

If the relevant random variable is clear from context, then the variance and standard deviation are often denoted by σ^2 and σ ("sigma"), just as the mean is μ ("mu").

What does this mean? First, let us rewrite the definition explicitly as a sum. If X takes values x_1, x_2, \dots, x_n with probability mass function, $p(x_i)$ then

$$\text{Var}(X) = E((X - \mu)^2) = \sum_{i=1}^n p(x_i) (x_i - \mu)^2.$$

In words, the formula for $\text{Var}(X)$ says to take a weighted average of the squared distance to the mean. By squaring, we make sure we are averaging only non-negative values, so that the spread to the right of the mean will not cancel that to the left.

Note on units:

1. σ has the same units as
2. $\text{Var}(X)$ has the same units as the square of X . So, if X is in meters, then $\text{Var}(X)$ is in meters squared.

Because σ and X have the same units, the standard deviation is a natural measure of spread.

Variance and Standard Deviation of a Discrete Random Variable



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Check Your Understanding: Variance and Standard Deviation for Discrete Random Variables



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BERNOULLI



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Bernoulli Distributions

Model: The Bernoulli distribution models one trial in an experiment that can result in either a success or failure. This is the most important distribution and is also the simplest. A random variable X has a Bernoulli distribution with parameter p if:

1. X takes the values 0 and 1.

$$2. P(X = 1) = p \text{ and } P(X = 0) = 1 - p$$

We will write $X \sim \text{Bernoulli}(p)$ or $\text{Ber}(p)$, which is read “ X follows a Bernoulli distribution with parameter p ” or “ X is drawn from a Bernoulli distribution with parameter p .”

A simple model for the Bernoulli distribution is to flip a coin with probability p of heads, with $X = 1$ on heads and $X = 0$ on tails. The general terminology is to say X is 1 on success and 0 on failure, with success and failure defined by the context.

Many decisions can be modeled as a binary choice, such as votes for or against a proposal. If p is the proportion of the voting population that favors the proposal, then the vote of a random individual is modeled by a Bernoulli (p).

Here are the table and graphs of the pmf for the Bernoulli ($1/2$) distribution and below that for the general Bernoulli (p) distribution.

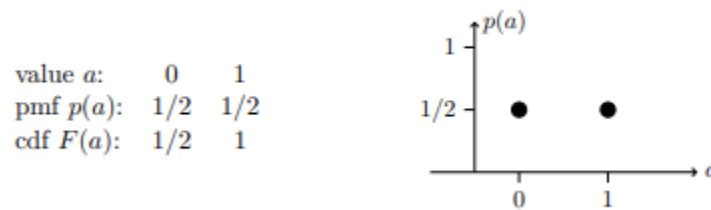


Figure 6: pmf for the Bernoulli($1/2$) distribution

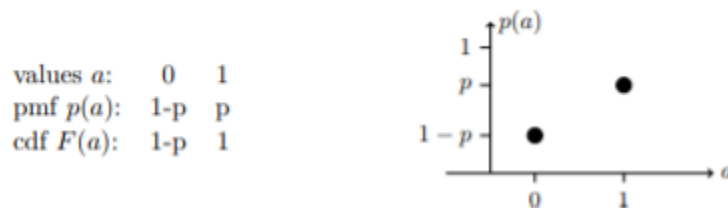


Figure 7: pmf for the Bernoulli(p) distribution

Mean and Variance of Bernoulli Distribution



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Bernoulli Distribution Mean and Variance





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The Variance of a Bernoulli Random Variable

Bernoulli random variables are fundamental, so we should know their variance.

If $X \sim \text{Bernoulli}(p)$ then

$$\text{Var}(X) = p(1 - p)$$

Proof: We know that $E(X) = p$. We compute $\text{Var}(X)$ using a table (Table 4).

Table 4

values X	0	1
pmf $p(x)$	$1 - p$	p
$(X - \mu)^2$	$(0 - p)^2$	$(1 - p)^2$

$$\text{Var}(X) = (1 - p)p^2 + p(1 - p)^2 = (1 - p)p(1 - p + p) = (1 - p)p$$

BINOMIAL DISTRIBUTIONS

Binomial Variables



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Check Your Understanding: Binomial Variables



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Binomial Distribution



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Visualizing a Binomial Distribution



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Binomial Distributions

The binomial distribution $\text{Binomial}(n, p)$, or $\text{Bin}(n, p)$, models the number of successes in n independent Bernoulli(p) trials.

There is a hierarchy here. A single Bernoulli trial is, say, a toss of a coin. A single binomial trial consists of n Bernoulli trials. For coin flips the sample space for a Bernoulli trial is $\{H, T\}$. The sample space for a binomial trial is all *sequences* of heads and tails of length n . Likewise, a Bernoulli random variable takes values 0 and 1 and a binomial random variable takes values 0, 1, 2, ..., n .

Example: The number of heads in n flips of a coin with probability p of heads follows a $\text{Binomial}(n, p)$ distribution.

We describe $X \sim \text{Binomial}(n, p)$ by giving its values and probabilities. For notation we will use k to mean an arbitrary number between 0 and n .

We remind you that ‘ n choose k ’ $= \binom{n}{k} = {}_n C_k$ is the number of ways to choose k things out of a collection of n things and it has the formula

$$\binom{n}{k} = {}_n C_k = \frac{n!}{k!(n-k)!}$$

(It is also called a binomial coefficient). Table 5 is a table for the pmf of a $\text{Binomial}(n, k)$ random variable:

Table 5

values a :	0	1	2	...	k	...	n
pmf $p(a)$:	$(1-p)^n$	$\binom{n}{1} p^1 (1-p)^{n-1}$	$\binom{n}{2} p^2 (1-p)^{n-2}$...	$\binom{n}{k} p^k (1-p)^{n-k}$...	p^n

Example: What is the probability of 3 or more heads in 5 tosses of a fair coin?

Answer: The binomial coefficients associated with $n = 5$ are

$$\binom{5}{0} = 1, \quad \binom{5}{1} = \frac{5!}{1!4!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 5, \quad \binom{5}{2} = \frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{5 \cdot 4}{2} = 10$$

And similarly

$$\binom{5}{3} = 10, \quad \binom{5}{4} = 5, \quad \binom{5}{5} = 1$$

Using these values, we get Table 6 for $X \sim \text{Binomial}(5, p)$.

Table 6

values a :	0	1	2	3	4	5
pmf $p(a)$:	$(1-p)^5$	$5p(1-p)^4$	$10p^2(1-p)^3$	$10p^3(1-p)^2$	$5p^4(1-p)$	p^5

We were told $p = 1/2$ so

$$P(X \geq 3) = 10 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 + 5 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^5 = \frac{16}{32} = \frac{1}{2}$$

Binomial Probability Example



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Explanation of the Binomial Probabilities

For concreteness, let $n = 5$ and $k = 2$ (the argument for arbitrary n and k is identical). So $X \sim \text{Binomial}(5, p)$ and we want to compute $p(2)$. The long way to compute $p(2)$ is to list all the ways to get exactly 2 heads in 5-coin flips and add up their probabilities. The list has 10 entries: HHTTT, HTHTT, HTTHT, HTTTH, THHTT, THTHT, THTTH, TTHHT, TTHTH, TTTHH.

Each entry has the same probability of occurring, namely

$$p^2(1-p)^3$$

This is because each of the two heads has probability p and each of the 3 tails has

probability $1 - p$. Because the individual tosses are independent, we can multiply probabilities. Therefore, the total probability of exactly 2 heads is the sum of 10 identical probabilities, i.e., $p(2) = 10p^2(1 - p)^3$, as shown in Table 6.

This guides us to the shorter way to do the computation. We have to count the number of sequences with exactly 2 heads. To do this we need to choose 2 of the tosses to be heads and the remaining 3 to be tails. The number of such sequences is the number of ways to choose 2 out of the 5 things, which is $\binom{5}{2}$. Since each such sequence has the same probability, $p^2(1 - p)^3$, we get the probability of exactly 2 heads $p(2) = \binom{5}{2} p^2(1 - p)^3$.

Here are some binomial probability mass functions (here, frequency is the same as probability).

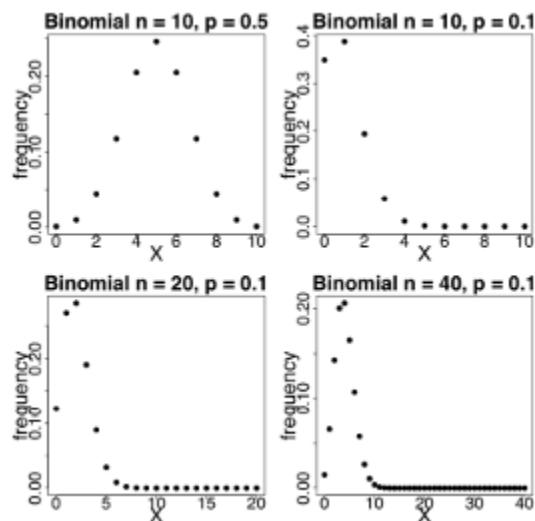


Figure 8

Check Your Understanding: Binomial Probability



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Characteristics of a Binomial Experiment

1. There are a fixed number of trials. Think of trials as repetitions of an

experiment. The letter n denotes the number of trials.

2. The random variable, x , number of successes, is discrete.
3. There are only two possible outcomes, called “success” and “failure,” for each trial. The letter p denotes the probability of a success on any one trial, and q denotes the probability of a failure on any one trial. $p + q = 1$
4. The n trials are independent and are repeated using identical conditions. Think of this as drawing WITH replacement. Because the n trials are independent, the outcome of one trial does not help in predicting the outcome of another trial. Another way of saying this is that for each individual trial, the probability, p , of a success and probability, q , of a failure remain the same. For example, randomly guessing at a true-false statistics question has only two outcomes. If a success is guessing correctly, then a failure is guessing incorrectly. Suppose Jow always guesses correctly on any statistics true-false question with a probability $p = 0.6$. Then, $q = 0.4$. This means that for every true-false statistics question Joe answers, his probability of success ($p = 0.6$) and his probability of failure ($q = 0.4$) remain the same.

Any experiment that has characteristics three and four and where $n = 1$ is called the Bernoulli Trial.

A word about independence

So far, we have been using the notion of an independent random variable without ever carefully defining it. For example, a binomial distribution is the sum of **independent** Bernoulli trials. This may (should?) have bothered you. Of course, we have an intuitive sense of what independence means for experimental trials. We also have the probabilistic sense that random variables X and Y are independent if knowing the value of X gives you no information about the value of Y .

Definition: The discrete random variables X and Y are independent if

$$P(X = a, Y = b) = P(X = a)P(Y = b)$$

For any values a, b . That is, the probabilities multiply.

Expected Value of a Binomial Variable



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Variance of a Binomial Variable



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Finding the Mean and Standard Deviation of a Binomial Random Variable



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Variance of Binomial (n, p)

Suppose $X \sim \text{Binomial}(n, p)$. Since X is the sum of *independent* Bernoulli(p) variables and each Bernoulli variable has variance $p(1 - p)$ we have

$$X \sim \text{Binomial}(n, p) \Rightarrow \text{Var}(X) = np(1 - p)$$

Check Your Understanding: Expected Value of a Binomial Variable



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GEOMETRIC DISTRIBUTIONS

Geometric Random Variables Introduction



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Check Your Understanding: Binomial vs. Geometric Random Variables



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Geometric Distributions

A **geometric distribution** models the number of tails before the first head in a sequence of coin flips (Bernoulli trials).

Example: (a) Flip a coin repeatedly. Let X be the number of tails before the first heads. So, X can equal 0, i.e., the first flip is heads, 1, 2, In principle, it takes any nonnegative integer value.

(b) Give a flip of tails the value 0 and heads the value 1. In this case, X is the number of 0's before the first 1.

(c) Give a flip of tails the value 1 and heads the value 0. In this case, X is the number of 1's before the first 0.

(d) Call a flip of tails a success and heads a failure. So, X is the number of successes before the first failure.

(e) Call a flip of tails a failure and heads a success. So, X is the number of failures before the first success.

You can see this models many different scenarios of this type. The most neutral language is the number of tails before the first head.

Formal definition: The random variable X follows a geometric distribution with parameter p if

- X takes the values 0, 1, 2, 3, ...
- Its pmf is given by $p(k) = P(X = k) = (1 - p)^k p$

We denote this by $X \sim \text{geometric}(p)$ or $\text{geo}(p)$. In Table 7 we have:

Table 7: $X \sim \text{geometric}(p)$: $X =$ the number of 0 s before the first 1.

value	$a:$	0	1	2	3	...	k	...
pmf	$p(a):$	p	$(1-p)p$	$(1-p)^2p$	$(1-p)^3p$...	$(1-p)^k p$...

The geometric distribution is an example of a discrete distribution that takes an infinite number of possible values. Things can get confusing when we work with successes and failures since we might want to model the number of successes before the first failure, or we might want the number of failures before the first success. To keep things straight you can translate to the neutral language of the number of tails before the first heads.

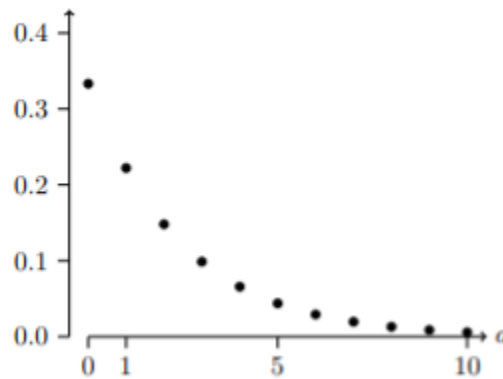


Figure 9: Pmf for the geometric(1/3) distribution

Probability for a Geometric Random Variable



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Characteristics of a Geometric Experiment

The geometric probability density function builds upon what we have learned from the binomial distribution. In this case, the experiment continues until either a success or a failure occurs rather than for a set number of trials. Here are the main characteristics of a geometric experiment:

1. There are one or more Bernoulli trials with all failures except the last one, which is a success. In other words, you keep repeating what you are doing until the first success. Then you stop. For example, you throw a dart at a bullseye until you hit the bullseye. The first time you hit the bullseye is a “success,” so you stop throwing the dart. It might take six tries until you hit the bullseye. You can think of the trials as failure, failure, failure, failure, failure, success, STOP.

2. In theory, the number of trials could go on forever.
3. The probability, p , of a success and the probability, q , of a failure is the same for each trial. $p + q = 1$ and $q = 1 - p$. For example, the probability of rolling a three when you throw one fair die is $\frac{1}{6}$. This is true no matter how many times you roll the die. Suppose you want to know the probability of getting the first three on the fifth roll. On rolls one through four, you do not get a face with a three. The probability for each of the rolls is $q = \frac{5}{6}$, the probability of a failure. The probability of getting a three on the fifth roll is $\left(\frac{5}{6}\right) \left(\frac{5}{6}\right) \left(\frac{5}{6}\right) \left(\frac{5}{6}\right) \left(\frac{1}{6}\right) = 0.0804$
4. X = the number of independent trials until the first success.

Check Your Understanding: Probability for a Geometric Random Variable



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Cumulative Geometric Probability (greater than a value)



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Cumulative Geometric Probability (less than a value)



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Check Your Understanding: Cumulative Geometric Probability



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POISSON DISTRIBUTIONS

Poisson Process 1



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Poisson Process 2



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Poisson Distribution

Another useful probability distribution is the Poisson distribution or waiting time distribution. This distribution is used to determine how many checkout clerks are needed to keep the waiting time in line to specified levels, how many telephone lines are needed to keep the system from overloading, and many other practical applications. The distribution gets its name from Simeon Poisson who presented it in 1837 as an extension of the binomial distribution.

Here are the main characteristics of a Poisson experiment:

1. The Poisson probability distribution gives the probability of a number of events occurring in a fixed interval of time or space if these events happen with a known average rate.
2. The events are independent of the time since the last event. For example, a book editor might be interested in the number of words spelled incorrectly in a particular book. It might be that, on the average, there are five words spelled incorrectly in 100 pages. The interval is the 100 pages, and it is assumed that

there is no relationship between when misspellings occur.

3. The random variable X = the number of occurrences in the interval of interest.

Example: You notice that a news reporter says “uh,” on average, two times per broadcast. What is the probability that the news reporter says “uh” more than two times per broadcast?

This is a Poisson problem because you are interested in knowing the number of times the news reporter says “uh” during a broadcast.

(a) What is the interval of interest?

- a. one broadcast measured in minutes

(b) What is the average number of times the news reporter says “uh” during one broadcast?

- a. 2

(c) Let $X = \underline{\hspace{1cm}}$. What values does X take on?

- a. Let X = the number of times the news reporter says “uh” during one broadcast.
 $x = 0, 1, 2, 3, \dots$

d. The probability question is $P(\underline{\hspace{1cm}})$.

- a. $P(x > 2)$

Notation for the Poisson: P = Poisson Probability Distribution Function

$$X \sim P(\mu)$$

Read this as “ X is a random variable with a Poisson distribution.” The parameter is μ (or λ); μ (or λ) = the mean for the interval of interest. The mean is the number of occurrences that occur on average during the interval period.

The formula for computing probabilities that are from a Poisson process is:

$$P(x) = \frac{\mu^x e^{-\mu}}{x!}$$

where $P(X)$ is the probability of X successes, μ is the expected number of successes based upon historical data, e is the natural logarithm approximately equal to 2.718, and X is the number of successes per unit, usually per unit of time.

In order to use the Poisson distribution, certain assumptions must hold. These are: the probability of a success, μ , is unchanged within the interval, there cannot be simultaneous

successes within the interval, and finally, that the probability of a success among intervals is independent, the same assumption of the binomial distribution.

In a way, the Poisson distribution can be thought of as a clever way to convert a continuous random variable, usually time, into a discrete random variable by breaking up time into discrete independent intervals. This way of thinking about the Poisson helps us understand why it can be used to estimate the probability for the discrete random variable from the binomial distribution. The Poisson is asking for the probability of a number of successes during a period of time while the binomial is asking for the probability of a certain number of successes for a given number of trials.

Example: Leah's answering machine receives about six telephone calls between 8 a.m. and 10 a.m. What is the probability that Leah receives more than one call in the next 15 minutes?

Let X = the number of calls Leah receives in 15 minutes (the interval of interest is 15 minutes or $\frac{1}{4}$ hour)

$x = 0, 1, 2, 3, \dots$

If Leah receives, on average, six telephone calls in two hours, and there are eight 15 minutes intervals in two hours, then Leah receives $(\frac{1}{8})(6) = 0.75$ calls in 15 minutes, on average. So, $\mu = 0.75$ for this problem.

$$X \sim P(0.75)$$

$$P(x > 1) = 0.1734$$

Probability that Leah receives more than one telephone call in the next 15 minutes is about 0.1734.

The graph of $X \sim P(0.75)$ is:

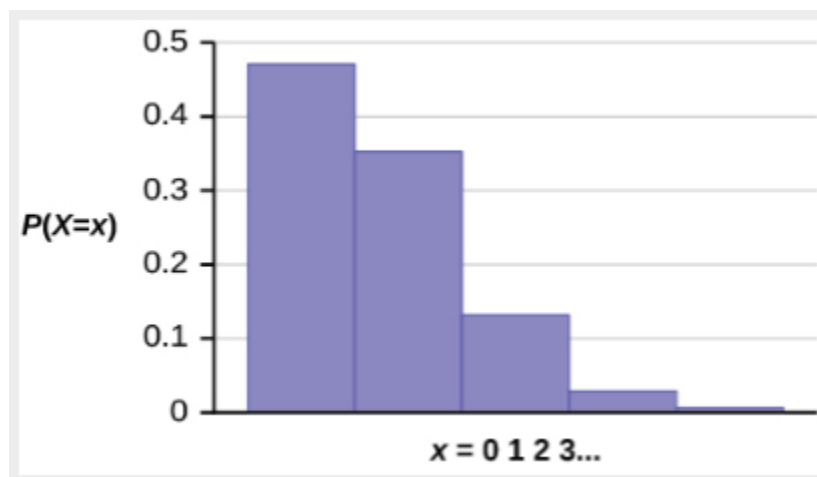


Figure 10

The y-axis contains the probability of x where X = the number of calls in 15 minutes.

Poisson Probability Distribution – Example



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Check Your Understanding: Poisson Distributions



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TABLES OF DISTRIBUTIONS AND PROPERTIES

Table 8

Distribution	Range	pmf	Mean	Variance
Bernoulli(p)	0,1	$p(0) = 1 - p, \quad p(1) = p$	p	$p(1 - p)$
Binomial(n, p)	$0, 1, \dots, n$	$p(k) = \binom{n}{k} p^k (1 - p)^{n-k}$	np	$np(1 - p)$
Geometric(p)	$0, 1, 2, \dots$	$p(k) = p(1 - p)^k$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$

Let X be a discrete random variable with range x_1, x_2, \dots and pmf $p(x_j)$

Table 9

Expected Value:	Variance:
Synonyms: mean, average	
Notation: $E(X), \mu$	$\text{Var}(X), \sigma^2$
Definition: $E(X) = \sum_j p(x_j) x_j$	$E((X - \mu)^2) = \sum_j p(x_j) (x_j - \mu)^2$

SET UP AND WORK WITH DISTRIBUTIONS FOR CONTINUOUS RANDOM VARIABLES

This section will explain how to work with distributions for continuous random variables. It will start with some definition and a calculus warm-up to help prepare you. The videos help explain the concepts while there are problems after each short section to help check your understanding.

CONTINUOUS RANDOM VARIABLES

We now turn to continuous random variables. All random variables assign a number to each outcome in a sample space. Whereas discrete random variables take on a discrete set of possible values, continuous random variables have a continuous set of values.

Computationally, to go from discrete to continuous we simply replace sums by integrals. It will help you to keep in mind that (informally) an integral is just a continuous sum.

Calculus Warmup

Conceptually, you should be comfortable with two views of a definite integral.

1. $\int_a^b f(x)dx = \text{area under the curve } y = f(x)$
2. $\int_a^b f(x)dx = \text{'sum of } f(x)dx'$

The connection between the two is seen below in Figure 11:

$$\text{area} \approx \text{sum of rectangle areas} = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x = \sum_{i=1}^n f(x_i)\Delta x.$$

Figure 11

As the width Δx of the intervals gets smaller the approximation becomes better.

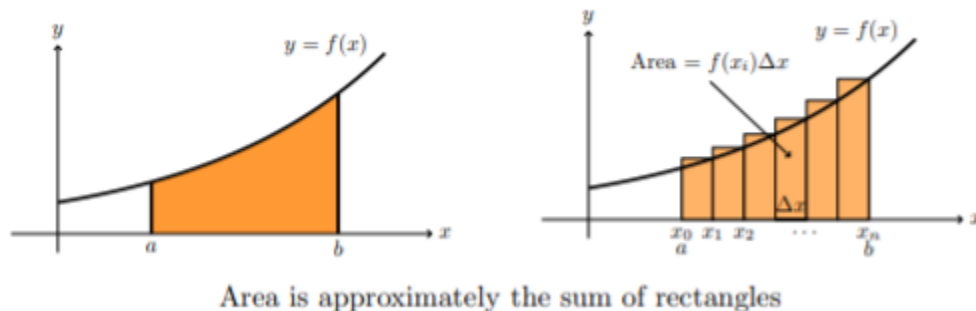


Figure 12

Note: In calculus, you learned to compute integrals by finding antiderivatives. This is important for calculations, but do not confuse this method for the reason we use integrals. Our interest in integrals comes primarily from its interpretation as a “sum” and to a much lesser extent its interpretation as area.

Probability Density Functions



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Continuous Random Variables and Probability Density Functions

A continuous random variable takes a range of values, which may be finite in extent. Here are a few examples of ranges: $[0, 1]$, $[0, \infty)$, $(-\infty, \infty)$, $[a, b]$.

Definition: A random variable X is continuous if there is a function $f(x)$ such that for any $c \leq d$ we have

$$P(c \leq X \leq d) = \int_c^d f(x)dx$$

The function $f(x)$ is called the **probability density function (pdf)**.

The pdf always satisfies the following properties:

1. $f(x) \geq 0$ (f is nonnegative).
2. $\int_{-\infty}^{\infty} f(x)dx = 1$ (This is equivalent to: $P(-\infty < X < \infty) = 1$)

The probability density function $f(x)$ of a continuous random variable is the analogue of the probability mass function $p(x)$ of a discrete random variable. Here are two significant differences:

1. Unlike $p(x)$, the pdf $f(x)$ is *not* a probability. You have to integrate it to get probability.
2. Since $f(x)$ is not a probability, there is no restriction that $f(x)$ be less than or equal to 1.

Note: In property 2, we integrated over $(-\infty, \infty)$ since we did not know the range of values taken by X . Formally, this makes sense because we just define $f(x)$ to be 0 outside of the range of X . In practice, we would integrate between bounds given by the range of X .

The Terms “Probability Mass” and “Probability Density”

Why do we use the terms mass and density to describe the pmf and pdf? What is the difference between the two? The simple answer is that these terms are completely analogous to the mass and density you saw in physics and calculus. We will review this first for the probability mass function and then discuss the probability density function. in extent.

Mass as a sum:

If masses m_1, m_2, m_3 , and m_4 are set in a row at positions x_1, x_2, x_3 , and x_4 , then the total mass is $m_1 + m_2 + m_3 + m_4$



Figure 13

We can define a ‘mass function’ $p(x)$ with $p(x_j) = m_j$ for $j = 1, 2, 3, 4$ and $p(x) = 0$ otherwise. In this notation the total mass is

$$p(x_1) + p(x_2) + p(x_3) + p(x_4)$$

The **probability mass function** behaves in exactly the same way, except it has the dimension of probability instead of mass.

Mass as an integral of density:

Suppose you have a rod of length L meters with varying density $f(x)$ kg/m. (Note the units are mass/length).

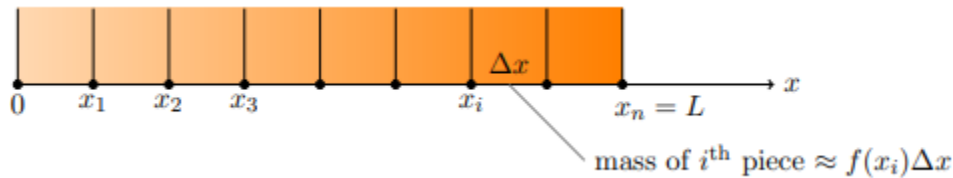


Figure 14

If the density varies continuously, we must find the total mass of the rod by integration:

$$\text{total mass} = \int_0^L f(x)dx.$$

This formula comes from dividing the rod into small pieces and ‘summing’ up the mass of each piece. That is: total mass $\approx \sum_{i=1}^n f(x_i) \Delta x$

In the limit as Δx goes to zero the sum becomes the integral.

The **probability density function** behaves exactly the same way, except it has units of probability/ (unit x) instead of kg/m. Indeed, the equation $P(c \leq X \leq d) = \int_c^d f(x)dx$ is exactly analogous to the above integral for total mass.

While we are on a physics kick, note that for both discrete and continuous random variables, the expected value is simply the center of mass or balance point.

Properties of Continuous Probability Density Functions

The graph of a continuous probability distribution is a curve. Probability is represented by area under the curve. The curve is called the **probability density function** (pdf). We use the symbol $f(x)$ to represent the curve. $f(x)$ is the function that corresponds to the graph; we use the density function $f(x)$ to draw the graph of the probability distribution.

Area under the curve is given by a different function called the **cumulative distribution function** (cdf). The cumulative distribution function is used to evaluate probability as area. Mathematically, the cumulative probability density function is the integral of the pdf, and the probability between two values of a continuous random variable will be the integral of the pdf between these two values: the area under the curve between these values. Remember that the area under the pdf for all possible values of the random

variable is one, certainty. Probability thus can be seen as the relative percent of certainty between the two values of interest.

- The outcomes are measured, not counted.
- The entire area under the curve and above the x-axis is equal to one.
- Probability is found for intervals of x values rather than for individual x values.
- $P(c < x < d)$ is the probability that the random variable X is in the interval between the values c and d . $P(c < x < d)$ is the area under the curve, above the x-axis, to the right of c and the left of d .
- $P(x = c) = 0$ The probability that x takes on any single individual value is zero. The area below the curve, above the x-axis, and between $x = c$ and $x = c$ has no width, and therefore no area (area = 0). Since the probability is equal to the area, the probability is also zero.
- $P(c < x < d)$ is the same as $P(c \leq x \leq d)$ because probability is equal to area.

We will find the area that represents probability by using geometry, formulas, technology, or probability tables. In general, integral calculus is needed to find the area under the curve for many probability density functions.

There are many continuous probability distributions. When using a continuous probability distribution to model probability, the distribution used is selected to model and fit the particular situation in the best way.

In this section, we will study the uniform distribution, the exponential distribution, and the normal distribution. The following graphs illustrate these distributions.

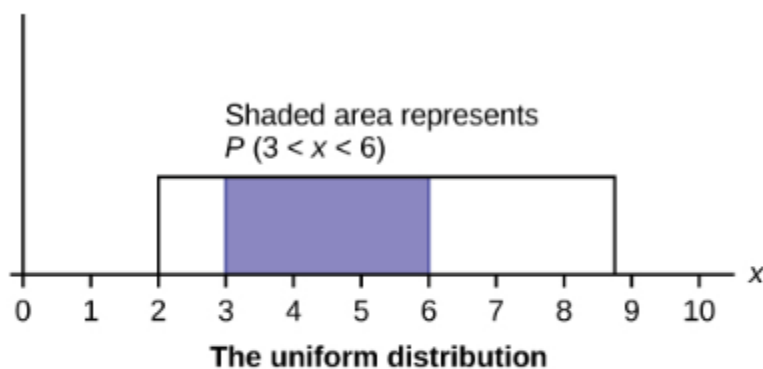


Figure 5.2 The graph shows a Uniform Distribution with the area between $x = 3$ and $x = 6$ shaded to represent the probability that the value of the random variable X is in the interval between three and six.

Figure 15

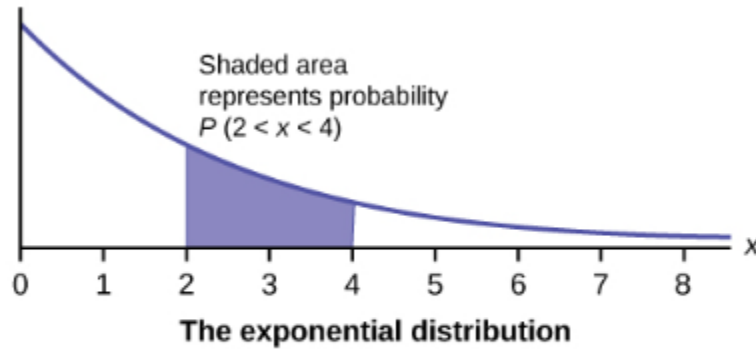


Figure 5.3 The graph shows an Exponential Distribution with the area between $x = 2$ and $x = 4$ shaded to represent the probability that the value of the random variable X is in the interval between two and four.

Figure 16

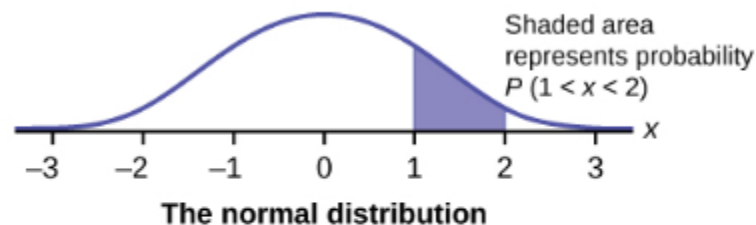


Figure 5.4 The graph shows the Standard Normal Distribution with the area between $x = 1$ and $x = 2$ shaded to represent the probability that the value of the random variable X is in the interval between one and two.

Figure 17

Cumulative Distribution Function

The cumulative distribution function (cdf) of a continuous random variable X is defined in exactly the same way as the cdf of a discrete random variable.

$$F(b) = P(X \leq b)$$

Note well that the definition is about probability. When using the cdf you should first think of it as a probability. Then when you go to calculate it, you can use

$$F(b) = P(X \leq b) = \int_{-\infty}^b f(x), \text{ where } f(x) \text{ is the pdf of } X$$

Notes:

1. For discrete random variables, there was not much occasion to use the cumulative distribution function. The cdf plays a far more prominent role for continuous random variables.

2. As before, we started the integral at $-\infty$ because we did not know the precise range of X . Formally, this still makes sense since $f(x) = 0$ outside the range of X . In practice, we will know the range and start the integral at the start of the range.
3. In practice, we often say “ X has distribution $F(x)$ ” rather than “ X has cumulative distribution function $F(x)$.”

Check Your Understanding: Continuous Random Variables



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UNIFORM DISTRIBUTIONS



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The Uniform Distribution

The uniform distribution is a continuous probability distribution and is concerned with events that are equally likely to occur. When working out problems that have a uniform distribution, be careful to not if the data is inclusive or exclusive of endpoints.

The mathematical statement of the uniform distribution is

$$f(x) = \frac{1}{b-a} \text{ for } a \leq x \leq b$$

Where a = the lowest value of x and b = the highest value of x .

Formulas for the theoretical mean and standard deviation are

$$\mu = \frac{a+b}{2} \text{ and } \sigma = \sqrt{\frac{(b-a)^2}{12}}$$

Uniform Distribution Properties

1. Parameters: a, b

2. Range: $[a, b]$
3. Notation: uniform (a, b) or $U(a, b)$
4. Density: $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$
5. Distribution: $F(x) = (x - a)/(b - a)$ for $a \leq x \leq b$
6. Models: All outcomes in the range have equal probability (more precisely all outcomes have the same probability density).

Graphs:

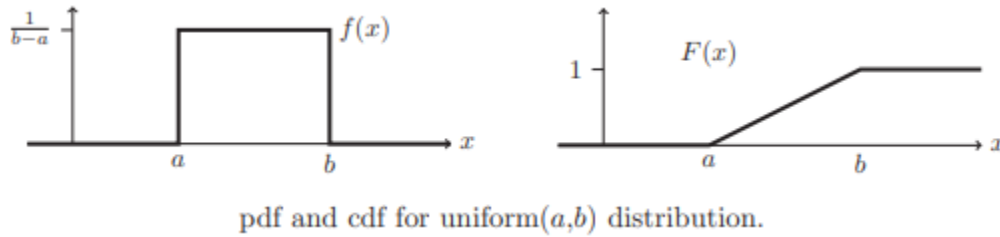


Figure 18

Examples:

1. Suppose we have a tape measure with markings at each millimeter. If we measure (to the nearest marking) the length of items that are roughly a meter long, the rounding error will uniformly distributed between -0.5 and 0.5 millimeters.
2. Many boardgames use spinning arrows (spinners) to introduce randomness. When spun, the arrow stops at an angle that is uniformly distributed between 0 and 2π radians.
3. In most pseudo-random number generators, the basic generator simulates a uniform distribution, and all other distributions are constructed by transforming the basic generator.

Check Your Understanding: Uniform Distributions



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EXPONENTIAL DISTRIBUTIONS



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The Exponential Distribution

The exponential distribution is often concerned with the amount of time until some specific occurs. For example, the amount of time (beginning now) until an earthquake occurs has an exponential distribution. Other examples include the length of time, in minutes, of long-distance business telephone calls, and the amount of time, in months, a car battery lasts. It can be shown, too, that the value of the change that you have in your pocket or purse approximately follows an exponential distribution.

Values for an exponential random variable occur in the following way. There are fewer large values and more small values. For example, marketing studies have shown that the amount of money customers spend in one trip to the supermarket follows an exponential distribution. There are more people who spend small amounts of money and fewer people who spend large amounts of money.

Exponential distributions are commonly used in calculations of product reliability, or length of time a produce lasts.

The random variable for the exponential distribution is continuous and often measures a passage of time, although it can be used in other applications. Typical questions may be, “what is the probability that some event will occur within the next x hours or days, or what is the probability that some event will occur between x_1 hours and x_2 hours, or what is the probability that the event will take more than x_1 hours to perform?” In short, the random variable X equals (a) the time between events or (b) the passage of time to complete an action, e.g., wait on a customer. The probability density function is given by:

$$f(x) = \frac{1}{\mu} e^{\left(-\frac{1}{\mu}\right)x}$$

Where μ is the historical average waiting time and has a mean and standard deviation of $1/\mu$.

In order to calculate probabilities for specific probability density functions, the cumulative density function is used. The cumulative density function (cdf) is simply the integral of the pdf and is:

$$F(x) = \int_0^x \left[\frac{1}{\mu} e^{-\frac{x}{\mu}} \right] = 1 - e^{-\frac{x}{\mu}}$$

Example



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Check Your Understanding: Exponential Probability Distribution



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Integration by Parts – Exponential Distribution



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Memorylessness of the Exponential Distribution

Say that the amount of time between customers for a postal clerk is exponentially distributed with a mean of two minutes. Suppose that five minutes have elapsed since the last customer arrived. Since an unusually long amount of time has now elapsed, it would seem to be more likely for a customer to arrive within the next minute. With the exponential distribution, this is not the case – the additional time spend waiting for the next customer does not depend on how much time has already elapsed since the last customer. This is referred to as the **memoryless property**. The exponential and geometric probability density functions are the only probability functions that have the memoryless property. Specifically, the **memoryless property** says that

$$P(X > r + t \mid X > r) = P(X > t) \text{ for all } r \geq 0 \text{ and } t \geq 0$$

For example, if five minutes have elapsed since the last customer arrives, then the probability that more than one minute will elapse before the next customer arrives is computed by using $r = 5$ and $t = 1$ in the foregoing equation.

$$P(X > 5 + 1 \mid X > 5) = P(X > 1) = e^{(-0.5)(1)} = 0.6065$$

This is the same probability as that of waiting more than one minute for a customer to arrive after the previous arrival.

Check Your Understanding: Memorylessness of the Exponential Distribution



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Relationship Between the Poisson and the Exponential Distribution

There is an interesting relationship between the exponential distribution and the Poisson distribution. Suppose that the time that elapses between two successive events follows the exponential distribution with a mean of μ units of time. Also assume that these times are independent, meaning that the time between events is not affected by the times between previous events. If these assumptions hold, then the number of events per unit of time follows a Poisson distribution with mean μ . Recall that if X has the Poisson distribution with mean μ , then

$$P(X = x) = \frac{\mu^x e^{-\mu}}{x!}$$

The formula for the exponential distribution: $P(X = x) = \frac{1}{\mu} e^{\left(-\frac{1}{\mu}\right)(x)}$. Where μ = average time between occurrences.

We see that the exponential is the cousin of the Poisson distribution, and they are linked through this formula. There are important differences that make each distribution relevant for different types of probability problems.

First, the Poisson has a discrete random variable, x , where time; a continuous variable is artificially broken into discrete pieces. We saw that the number of occurrences of an event in a given time interval, x , follows the Poisson distribution.

For example, the **number** of times the telephone rings per hour. By contrast, the time **between** occurrences follows the exponential distribution. For example, the telephone just rang, how long will it be until it rings again? We are measuring length of time of the interval, a continuous random variable, exponential, not events during an interval, Poisson.

The Exponential Distribution v. the Poisson Distribution

A visual way to show both the similarities and differences between these two distributions is with a timeline.

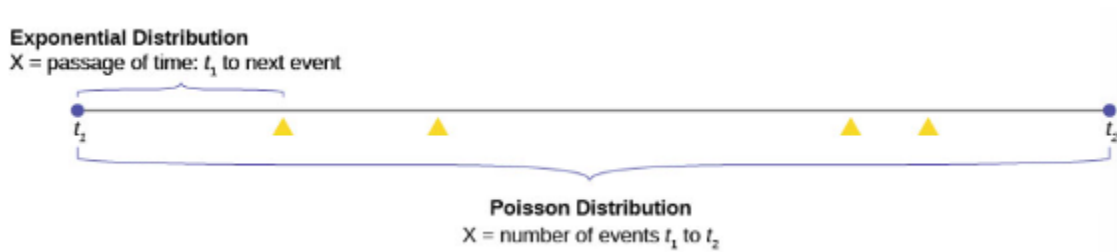


Figure 19

The random variable for the Poisson distribution is discrete and thus counts events during a given time period, t_1 to t_2 on Figure 19 above and calculates the probability of that number occurring. The number of events, four in the graph is measured in counting numbers; therefore, the random variable of the Poisson is a discrete random variable.

The exponential probability distribution calculates probabilities of the passage of time, a continuous random variable. In Figure 19, this is shown as the bracket from t_1 to the next occurrence of the event marked with a triangle.

Classic Poisson distribution questions are “how many people will arrive at my checkout window in the next hour?”

Classic exponential distribution questions are “how long it will be until the next person arrives,” or a variant, “how long will the person remain here once they have arrived?”

Again, the formula for the exponential distribution is:

$$f(x) = \frac{1}{\mu} e^{\left(-\frac{1}{\mu}\right)(x)}$$

We see immediately the similarity between the exponential formula and the Poisson formula.

$$P(x) = \frac{\mu^x e^{-\mu}}{x!}$$

Both probability density functions are based upon the relationship between time and exponential growth or decay. The “e” in the formula is a constant with the approximate value of 2.71828 and is the base of the natural logarithmic exponential growth formula. When people say that something has grown exponentially this is what they are talking about.

An example of the exponential and the Poisson will make clear the differences between the two. It will also show the interesting applications they have.

Poisson Distribution

Suppose that historically 10 customers arrive at the checkout lines each hour. Remember

that this is still probability, so we have to be told these historical values. We see this is a Poisson probability problem.

We can put this information into the Poisson probability density function and get a general formula that will calculate the probability of any specific number of customers arriving in the next hour.

The formula is for any value of the random variable we chose, and so the x is put into the formula. This is the formula:

$$f(x) = \frac{10^x e^{-10}}{x!}$$

As an example, the probability of 15 people arriving at the checkout counter in the next hour would be

$$P(x = 15) = \frac{10^{15} e^{-10}}{15!} = 0.0611$$

Here we have inserted $x = 15$ and calculated the probability that in the next hour 15 people will arrive is .061.

Exponential Distribution

If we keep the same historical facts that 10 customers arrive each hour, but we now are interested in the service time a person spends at the counter, then we would use the exponential distribution. The exponential probability function for any value of x , the random variable, for this particular checkout counter historical data is:

$$f(x) = \frac{1}{.1} e^{-\frac{x}{.1}} = 10e^{-10x}$$

To calculate μ , the historical average service time, we simply divide the number of people that arrive per hour, 10, into the time period, one hour, and have $\mu = 0.1$. Historically, people spend 0.1 of an hour at the checkout counter, or 6 minutes. This explains the .1 in the formula.

There is a natural confusion with μ in both the Poisson and exponential formulas. They have different meanings, although they have the same symbol. The mean of the exponential is one divided by the mean of the Poisson. If you are given the historical number of arrivals, you have the mean of the Poisson. If you are given an historical length of time between events, you have the mean of an exponential.

Continuing with our example at the checkout clerk; if we wanted to know the probability that a person would spend 9 minutes or less checking out, then we use this formula. First, we convert to the same time units which are parts of one hour. Nine minutes is 0.15 of one hour. Next, we note that we are asking for a range of values. This is always the case for a continuous random variable. We write the probability question as:

$$p(x \leq 9) = 1 - 10e^{-10x}$$

We can now put the numbers into the formula, and we have our result

$$p(x = .15) = 1 - 10e^{(-10)(.15)} = 0.7769$$

The probability that a customer will spend 9 minutes or less checking out is 0.7769.

We see that we have a high probability of getting out in less than nine minutes and a tiny probability of having 15 customers arriving in the next hour.

Exponential Distribution Properties

1. Parameter: $\lambda, \left(\frac{1}{\mu}\right)$
2. Range: $[0, \infty)$
3. Notation: exponential (λ) or $\exp(\lambda)$
4. Density: $f(x) = \lambda e^{-\lambda x}$ or $f(x) = \frac{1}{\mu} e^{\left(-\frac{1}{\mu}\right)(x)}$ for $0 \leq x$
5. Distribution: (easy integral) $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$
6. Right tail distribution: $P(X > x) = 1 - F(x) = e^{-\lambda x}$
7. Models: The waiting time for a continuous process to change state.

Examples:

1. If I step out to 77 Mass Ave after class and wait for the next taxi, my waiting time in minutes is exponentially distributed. We will see that in this case λ is given by one over the average number of taxis that pass per minute (on weekday afternoons).
2. The exponential distribution models the waiting time until an unstable isotope undergoes nuclear decay. In this case, the value of λ is related to the half-life of the isotope.

Graphs:

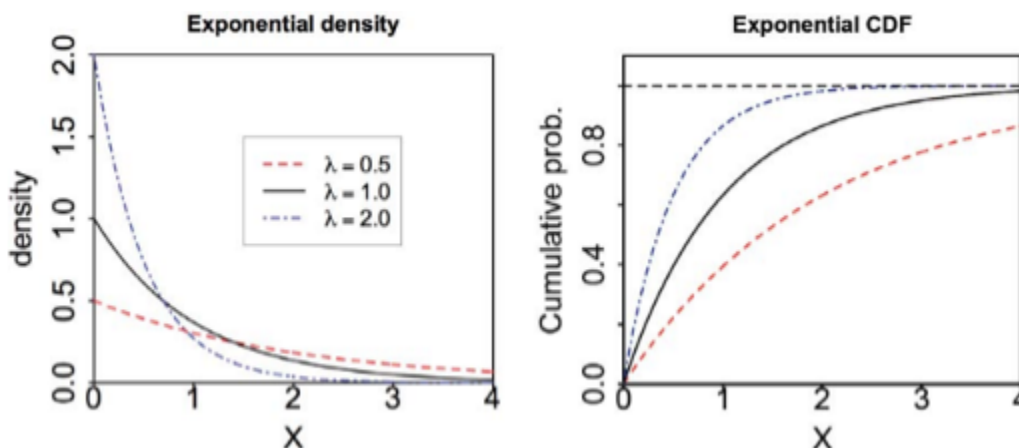


Figure 20

THE NORMAL DISTRIBUTION

The normal probability density function, a continuous distribution, is the most important of all the distributions. It is widely used and even more widely abused. Its graph is bell-shaped. You see the bell curve in almost all disciplines. Some of these include psychology, business, economics, the sciences, nursing, and, of course, mathematics. Some of your instructors may use the normal distribution to help determine your grade. Most IQ scores are normally distributed. Often real-estate prices fit a normal distribution.

The normal distribution has two parameters (two numerical descriptive measures): the mean (μ) and the standard deviation (σ). If X is a quantity to be measured that has a normal distribution with mean (μ) and standard deviation (σ), we designate this by writing the following formula of the normal probability density function:

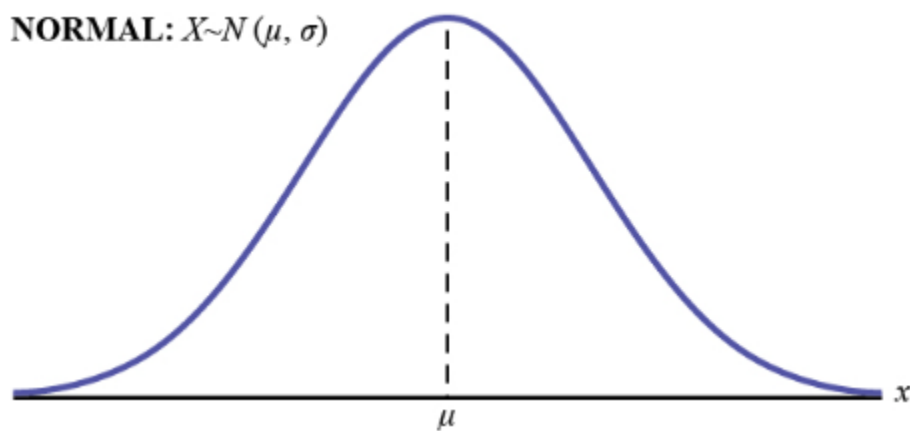


Figure 21

The probability density function is a rather complicated function. Do not memorize it. It is not necessary.

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2 \cdot \pi}} \cdot e^{-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2}$$

The curve is symmetric about a vertical line drawn through the mean, μ . The mean is the same as the median, which is the same as the mode, because the graph is symmetric about μ . As the notation indicates, the normal distribution depends only on the mean and the standard deviation. Note that this is unlike several probability density functions we have already studied, such as the Poisson, where the mean is equal to μ and the standard deviation simply the square root of the mean, or the binomial, where p is used to determine both the mean and standard deviation. Since the area under the curve must equal one, a change in the standard deviation, σ , causes a change in the shape of the normal curve; the curve becomes fatter and wider or skinnier and taller depending on σ . A change in μ causes the graph to shift to the left or right. This means there are an infinite number of normal probability distributions. One of the special interests is called the **standard normal distribution**.

The Standard Normal Distribution

The **standard normal distribution** is a normal distribution of **standardized values called z—scores**. A **z-score is measure in units of the standard deviation**.

The mean for the standard normal distribution is zero, and the standard deviation is one. What this does is dramatically simplify the mathematical calculation of probabilities. Take a moment and substitute zero and one in the appropriate places in the above formula and you can see that the equation collapses into one that can be much more easily solved using integral calculus. The transformation $z = \frac{x-\mu}{\sigma}$ produces the distribution $Z \sim N(0, 1)$. The value x in the given equation comes from a known normal distribution with known mean μ and known standard deviation σ . The z-score tells how many standard deviations a particular x is away from the mean.

Z-Scores

If X is a normally distributed random variable and $X \sim N(\mu, \sigma)$, then the z-score for a particular x is:

$$z = \frac{x-\mu}{\sigma}$$

The z-score tells you how many standard deviations the value x is above (to the right of) or below (to the left of) the mean, μ . Values of x that are larger than the mean have positive z-scores, and values of x that are smaller than the mean have negative z-scores. If x equals the mean, then x has a z-score of zero.

Example: Suppose $X \sim N(5, 6)$. This says that X is a normally distributed random variable with mean $\mu = 5$ and standard deviation $\sigma = 6$. Suppose $x = 17$. Then:

$$z = \frac{x-\mu}{\sigma} = \frac{17-5}{6} = 2$$

This means that $x = 17$ is **two standard deviations** (2σ) above or to the right of the mean $\mu = 5$.

Now suppose $x = 1$. Then: $z = \frac{x-\mu}{\sigma} = \frac{1-5}{6} = -0.67$ (rounded to two decimal places).

This means that $x = 1$ is 0.67 standard deviations (-0.67σ) below or to the left of the mean $\mu = 5$.

The Empirical Rule

If X is a random variable and has a normal distribution with mean μ and standard deviation σ , then the **Empirical Rule** states the following:

- About 68% of the x values lie between -1σ and $+1\sigma$ of the mean μ (within one standard deviation of the mean).
- About 95% of the x values lie between -2σ and $+2\sigma$ of the mean μ (within two standard deviations of the mean).
- About 99.7% of the x values lie between -3σ and $+3\sigma$ of the mean μ (within three standard deviations of the mean). Notice that almost all the x values lie within three standard deviations of the mean.
- The z -scores for $+1\sigma$ and -1σ are $+1$ and -1 respectively
- The z -scores for $+2\sigma$ and -2σ are $+2$ and -2 respectively.
- The z -scores for $+3\sigma$ and -3σ are $+3$ and -3 respectively.

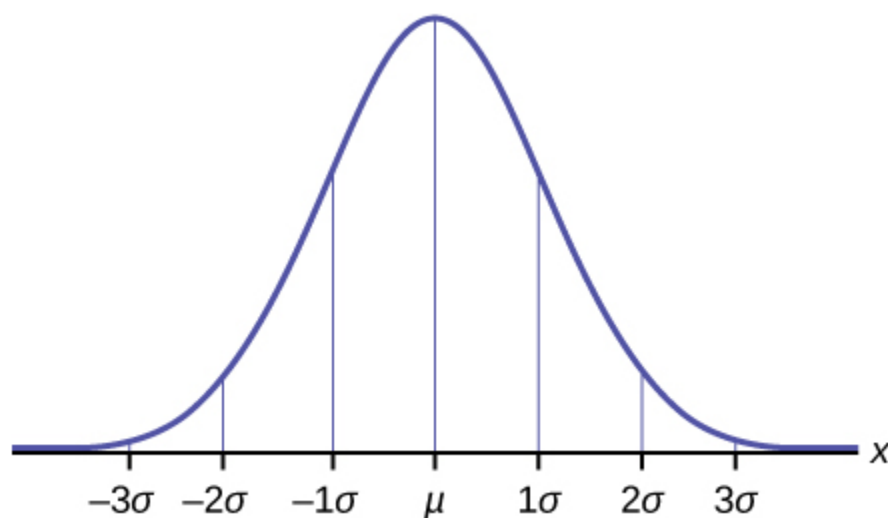


Figure 22

Normal Distribution Problems: Empirical Rule



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Check Your Understanding: Normal Distribution Problems



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Normal Distribution Properties

In 1809, Carl Friedrich Gauss published a monograph introducing several notions that have become fundamental to statistics: the normal distribution, maximum likelihood estimation, and the method of least squares. For this reason, the normal distribution is also called the *Gaussian* distribution, and it is the most important continuous distribution.

1. Parameters: μ, σ
2. Range: $(-\infty, \infty)$
3. Notation: normal (μ, σ^2) or $N(\mu, \sigma^2)$
4. Density: $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$
5. Distribution: $F(x)$ has no formula, so use tables or software such as “pnorm” in R to compute $F(x)$. (Will be discussed later in the module)
6. Models: Measurement error, intelligence/ability, height, averages of lots of data.

Here are some graphs of normal distributions. Note they are shaped like a *bell curve*. Note also that as σ increases as they become more spread out.

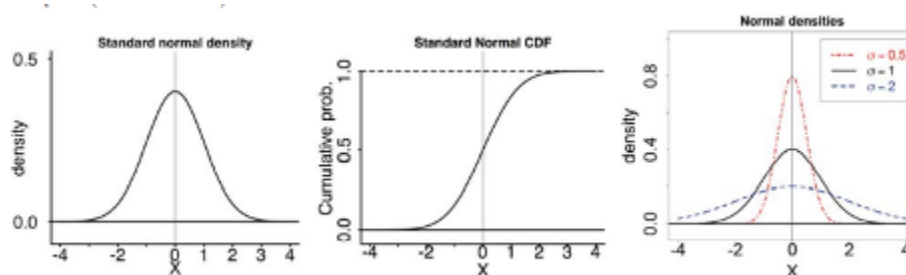


Figure 23

Using the Normal Distribution

The shaded area in Figure 24 indicates the area to the right of x . This area is represented by the probability $P(X > x)$. Normal tables provide the probability between the mean, zero for the standard normal distribution, and a specific value such as x_1 . This is the unshaded part of the graph from the mean to x_1 .

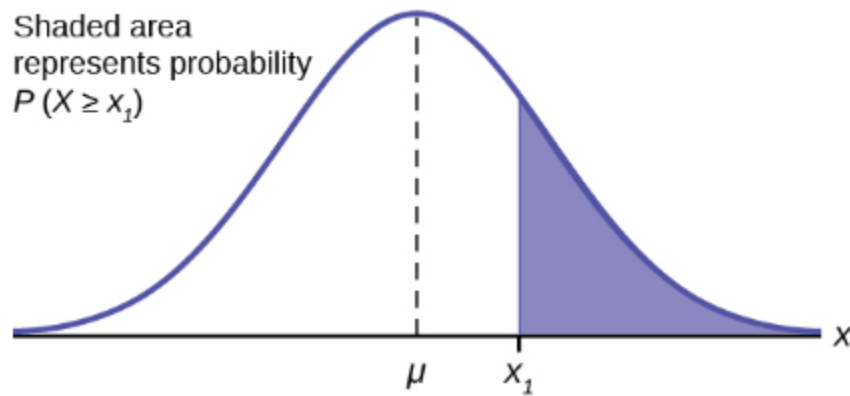


Figure 24

Because the normal distribution is symmetrical, if x_1 were the same distance to the left of the mean the area, probability, in the left tail, would be the same as the shaded area in the right tail. Also, bear in mind that because of the symmetry of this distribution, one-half of the probability is to the right of the mean and one-half is to the left of the mean.

Calculations of Probabilities

To find the probability for probability density functions with a continuous random variable we need to calculate the area under the function across the values of X we are interested in. For the normal distribution this seems a difficult task given the complexity of the formula. There is, however, a simple way to get what we want. Here again is the formula for the normal distribution:

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2 \cdot \pi}} \cdot e^{-\frac{1}{2} \cdot \left(\frac{x-\mu}{\sigma}\right)^2}$$

Looking at the formula for the normal distribution it is not clear just how we are going to solve for the probability doing it the same way we did it with the previous probability functions. There we put the data into the formula and did the math.

To solve this puzzle, we start knowing that the area under a probability density function is the probability.

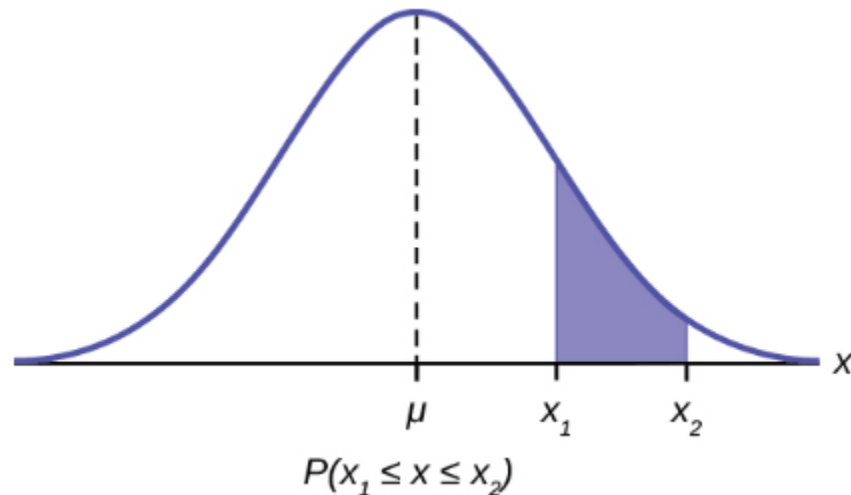


Figure 25

This shows that the area between X_1 and X_2 is the probability as stated in the formula:
 $P(X_1 \leq x \leq X_2)$

The mathematical tool needed to find the area under a curve is integral calculus. The integral of the normal probability density function between the two points x_1 and x_2 is the area under the curve between these two points and is the probability between these two points.

Doing these integrals is no fun and can be very time consuming. But now, remembering that there are an infinite number of normal distributions out there, we can consider the one with a mean of zero and a standard deviation of 1. This particular normal distribution is given the name Standard Normal Distribution. Putting these values into the formula it reduces to a very simple equation. We can now quite easily calculate all probabilities for any value of x , for this particular normal distribution, that has a mean of zero and a standard deviation of 1. They are presented in numerous ways. The table is the most common presentation and is set up with probabilities for one-half the distribution beginning with zero, the mean, and moving outward. The shaded area in the graph at the top of the table in Statistical Tables represents the probability from zero to the specific Z value noted on the horizontal axis, Z .

The only problem is that even with this table, it would be a ridiculous coincidence that our data had a mean of zero and a standard deviation of one. The solution is to convert the distribution we have with its mean and standard deviation to this new Standard Normal Distribution. The Standard Normal has a random variable called Z .

Using the standard normal table, typically called the normal table, to find the probability of one standard deviation, go to the Z column, reading down to 1.0 and then read at column 0. That number, 0.3413 is the probability from zero to 1 standard deviation. At the top of the table is the shaded area in the distribution which is the probability for one standard deviation. The table has solved our integral calculus problem. But only if our data has a mean of zero and a standard deviation of 1.

However, the essential point here is, the probability for one standard deviation on one normal distribution is the same on every normal distribution. If the population data set has a mean of 10 and a standard deviation of 5 then the probability from 10 to 15, one standard deviation, is the same as from zero to 1, one standard deviation on the standard normal distribution. To compute probabilities, areas, for any normal distribution, we need only to convert the particular normal distribution to the standard normal distribution and look up the answer in the tables. As review, here again is the standardizing formula:

$$Z = \frac{x - \mu}{\sigma}$$

where Z is the value on the standard normal distribution, X is the value from a normal distribution one wishes to convert to the standard normal, μ and σ are, respectively, the mean and standard deviation of that population. Note that the equation uses μ and σ which denotes population parameters. This is still dealing with probability, so we always are dealing with the population, with known parameter values and a known distribution. It is also important to note that because the normal distribution is symmetrical it does not matter if the z -score is positive or negative when calculating a probability. One standard deviation to the left (negative Z -score) covers the same area as one standard deviation to the right (positive Z -score). This fact is why the Standard Normal tables do not provide areas for the left side of the distribution. Because of this symmetry, the Z -score formula is sometimes written as:

$$Z = \frac{|x - \mu|}{\sigma}$$

Where the vertical lines in the equation means the absolute value of the number. What the standardizing formula is really doing is computing the number of standard deviations X is from the mean of its own distribution.

Check Your Understanding: Using the Normal Distribution



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Probabilities from Density Curves



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Check Your Understanding: Probabilities from Density Curves



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Standard Normal Table for Proportion Below



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Standard Normal Table for Proportion Above



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Check Your Understanding: Standard Normal Table for Proportion Below and Above





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Standard Normal Table for Proportion Between Values



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Check Your Understanding: Standard Normal Table for Proportion Between Values



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Finding z-score for a Percentile



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Check Your Understanding: Finding z-score for a Percentile



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EXPECTATION, VARIANCE AND STANDARD DEVIATION FOR CONTINUOUS RANDOM VARIABLES

So far, we have looked at expected value, standard deviation, and variance for discrete random variables. These summary statistics have the same meaning for continuous random variables:

- The expected value $\mu = E(X)$ is a measure of location or central tendency.
- The standard deviation σ is a measure of the spread or scale.
- The variance $\sigma^2 = \text{Var}(X)$ is the square of the standard deviation.

To move from discrete to continuous, we will simply replace the sums in the formulas by integrals.

Expected Value of a Continuous Random Variable

Definition: Let X be a continuous random variable with range $[a, b]$ and probability density function $f(x)$. The expected value of X is defined by

$$E(X) = \int_a^b x f(x) dx$$

Let us see how this compares with the formula for a discrete random variable:

$$E(X) = \sum_{i=1}^n x_i p(x_i)$$

The discrete formula says to take a weighted sum of the values x_i of X , where the weights are the probabilities $p(x_i)$. Recall that $f(x)$ is a probability density. Its units are prob/(unit of X).

So, $f(x)dx$ represents the probability that X is in an infinitesimal range of width dx around x . Thus, we can interpret the formula for $E(X)$ as a weighted integral of the values x of X , where the weights are the probabilities $f(x)dx$.

As before, the expected value is also called the mean or average.

Example 1: Let $X \sim \text{uniform}(0, 1)$. Find $E(X)$.

Answer: X has range $[0, 1]$ and density $f(x) = 1$. Therefore,

$$E(X) = \int_0^1 x dx = \left. \frac{x^2}{2} \right|_0^1 = \frac{1}{2}$$

Not surprisingly the mean is at the midpoint of the range.

Example 2: Let X have a range $[0, 2]$ and density $\frac{3}{8}x^2$. Find $E(X)$.

$$\text{Answer: } E(X) = \int_0^2 x f(x) dx = \int_0^2 \frac{3}{8} x^3 dx = \left. \frac{3x^4}{32} \right|_0^2 = \frac{3}{2}$$

Does it make sense that this X has mean in the right half of its range?

Yes. Since the probability density increases as x increases over the range, the average value of x should be in the right half of the range.

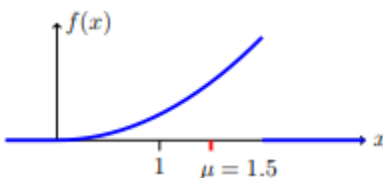


Figure 26

μ is “pulled” to the right of the midpoint 1 because there is more mass to the right.

Example 3: Let $X \sim \text{exp}(\lambda)$. Find $E(X)$.

Answer: The range of X is $[0, \infty)$ and its pdf is $f(x) = \lambda e^{-\lambda x}$.

$$E(X) = \int_0^\infty \lambda x e^{-\lambda x} dx = -x e^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda} \Big|_0^\infty = \frac{1}{\lambda}$$

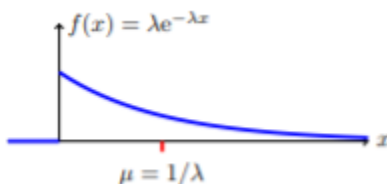


Figure 27: Mean of an exponential random variable

Example 4: Let $Z \sim N(0, 1)$. Find $E(Z)$.

Answer: The range of Z is $(-\infty, \infty)$ and its pdf is $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$$E(Z) = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} z e^{-\frac{z^2}{2}} dz = -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Big|_{-\infty}^\infty = 0$$

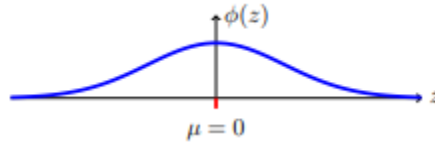


Figure 28: The standard normal distribution is symmetric and has mean 0.

Properties of $E(X)$

The properties of $E(X)$ for continuous random variables are the same as for discrete ones:

1. If X and Y are random variables on a sample space, Ω then

$$E(X + Y) = E(X) + E(Y)$$
2. If a and b are constants, then $E(aX + B) = aE(X) + b$

Example 5: In this example we verify that for $X \sim N(\mu, \sigma)$ we have $E(X) = \mu$.

Answer: Example (4) showed that for standard normal Z , $E(Z) = 0$. We could mimic the calculation there to show that $E(X) = \mu$. Instead, we will use the linearity properties of $E(X)$. $X \sim N(\mu, \sigma^2)$ is a normal random variable we can standardize it:

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

Inverting this formula, we have $X = \sigma Z + \mu$. The linearity of expected value now gives

$$E(X) = E(\sigma Z + \mu) = \sigma E(Z) + \mu = \mu$$

Expectation of Functions of X

This works exactly the same as the discrete case. If $h(x)$ is a function, then $Y = h(X)$ is a random variable and

$$E(Y) = E(h(X)) = \int_{-\infty}^{\infty} h(x) f_X(x) dx$$

Example 6: Let $X \sim \exp(\lambda)$. Find $E(X^2)$

Answer: Using integration by parts we have

$$\begin{aligned} E(X^2) &= \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \left[-x^2 e^{-\lambda x} - \frac{2x}{\lambda} e^{-\lambda x} - \frac{2}{\lambda^2} e^{-\lambda x} \right]_0^{\infty} \\ &= \frac{2}{\lambda^2} \end{aligned}$$

Variance

Now that we have defined expectation for continuous random variable, the definition of variance is identical to that of discrete random variables.

Definition: Let X be a continuous random variable with mean μ . The variance of X is

$$\text{Var}(X) = E((X - \mu)^2)$$

Properties of Variance

These are exactly the same as in the discrete case.

1. If X and Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$
2. For constants a and b , $\text{Var}(aX + b) = a^2 \text{Var}(X)$.
3. Theorem: $\text{Var}(X) = E(X^2) - E(X)^2 = E(X^2) - \mu^2$

For Property 1, note carefully the requirement that X and Y are independent.

Property 3 gives a formula for $\text{Var}(X)$ that is often easier to use in hand calculations.

Example 7: Let $X \sim \text{uniform}(0, 1)$. Find $\text{Var}(X)$ and σ_X

Answer: In Example 1 we found $\mu = \frac{1}{2}$. Next, we compute

$$\text{Var}(X) = E((X - \mu)^2) = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{12}$$

Example 8: Let $X \sim \exp(\lambda)$. Find $\text{Var}(X)$ and σ_X

Answer: In Examples 3 and 6 we computed

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda} \text{ and } E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}$$

So, by Property 3,

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \text{ and } \sigma_X = \frac{1}{\lambda}$$

We could have skipped Property 3 and computed this directly from

$$\text{Var}(X) = \int_0^\infty \left(x - \frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x} dx$$

Example 9: Let $Z \sim N(0, 1)$. Show $\text{Var}(Z) = 1$

Note: The notation for normal variables is $X \sim N(\mu, \sigma^2)$. This is certainly suggestive, but as mathematicians we need to prove that $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$. Above we showed $E(X) = \mu$. This example shows that $\text{Var}(Z) = 1$, just as the notation suggests. In the next example we will show $\text{Var}(X) = \sigma^2$

Answer: Since $E(Z) = 0$, we have

$$\text{Var}(Z) = E(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz$$

(Using integration by parts with $u = z, v' = ze^{-\frac{z^2}{2}}$

$$\Rightarrow u' = 1, v = -e^{-\frac{z^2}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \left(-ze^{-\frac{z^2}{2}} \Big|_{-\infty}^{\infty} \right) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

The first term equals 0 because the exponential goes to zero much faster than z grows at both $\pm\infty$. The second term equals 1 because it is exactly the total probability integral of the pdf $\varphi(z)$ for $N(0, 1)$. So $\text{Var}(X) = 1$.

Example 10: Let $X \sim N(\mu, \sigma^2)$. Show $\text{Var}(X) = \sigma^2$

Answer: This is an exercise in change of variables. Letting $z = (x - \mu)/\sigma$, we have

$$\begin{aligned} \text{Var}(X) &= E((X - \mu)^2) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x - \mu)^2 e^{-(x-\mu)^2/2\sigma^2} dx \\ &= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz = \sigma^2 \end{aligned}$$

The integral in the last line is the same one we computed for $\text{Var}(Z)$.

Quantiles

Definition: The median of X is the value x for which $P(X \leq x) = 0.5$, i.e. the value of x such that $P(X \leq X) = P(X \geq x)$. In other words, X has equal probability of being above or below the median, and each probability is therefore $1/2$. In terms of the cdf $F(x) = P(X \leq x)$, we can equivalently define the median as the value x satisfying $F(x) = 0.5$.

Think: What is the median of Z ?

Answer: By symmetry, the median is 0.

Example 11: Find the median of $X \sim \exp(\lambda)$.

Answer: The cdf of X is $F(x) = 1 - e^{-\lambda x}$. So, the median is the value of x for which $F(x) = 1 - e^{-\lambda x} = 0.5$. Solving for x we find: $x = (\ln 2)/\lambda$.

Think: In this case the median does not equal the mean of $\mu = 1/\lambda$. Based on the graph of the pdf of X can you argue why the median is to the left of the mean.

Definition: The p^{th} quantile of X is the value q_p such that $P(X \leq q_p) = p$

Notes:

1. In this notation the median is $q_{0.5}$
2. We will usually write this in terms of the cdf: $F(q_p) = p$

With respect to the pdf $f(x)$, the quantile q_p is the value such that there is an area of p to the left of q_p and an area of $1 - p$ to the right of q_p . In the examples below, note how we can represent the quantile graphically using either area of the pdf or height of the cdf.

Example 12: Find the 0.6 quantile for $X \sim U(0, 1)$.

Answer: The cdf for X is $F(x) = x$ on the range $[0, 1]$. So $q_{0.6} = 0.6$

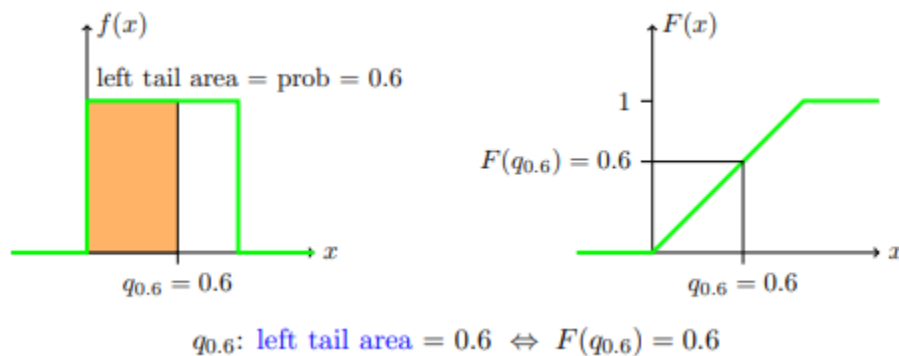


Figure 29

Quantile give a useful measure of location for a random variable.

Percentiles, decile, quartiles

For convenience, quantiles are often described in terms of percentile, decile, or quantiles. The 60th percentile is the same as the 0.6 quantile. For example, you are in the 60th percentile for height if you are taller than 60 percent of the population, i.e., the probability that you are taller than a randomly chosen person is 60 percent.

Likewise, deciles represent steps of $1/10$. The third decile is the 0.3 quantile. Quartiles are in steps of $1/4$. Third quartile is the 0.75 quantile and the 75th percentile.

RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section will relate the fundamentals of transportation engineering by discussion and showing you queuing theory and models to help your understanding.

QUEUING THEORY: EXPONENTIAL DISTRIBUTION



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POISSON MODEL OF VEHICLE ARRIVALS

The Poisson distribution is used to represent arrival patterns at a roadway location to conduct traffic queue and delay analysis. The distribution helps estimate the likelihood of n vehicles arriving in a specific time frame and the average arrival rate using data collected from roadway locations. For more information on the Poisson distribution, the reader is referred to in the above section labeled “Poisson Distributions.”

DISTRIBUTION OF TIME-MEAN SPEED

Spot speed studies during relatively congestion-free durations are conducted to estimate mean speed, modal speed, pace, standard deviation, and different percentile of speeds at a roadway location. 85th percentile speed estimated from a spot speed study is one of the crucial measures for determining the appropriate speed limit for the roadway section. Speed data for automobiles on roadway locations collected as part of the spot speed study typically follow the normal distribution discussed in “The Normal Distribution” section above.

TRAFFIC SAFETY

Traffic safety analysis involves statistical methods that are used to process records of historical collisions or crash data. The negative binomial distribution is used to describe crash occurrences on roadway segments and intersections. The negative binomial model’s formulation is a variant of Poisson’s distribution (see “Poisson Distributions” above) that allows the standard deviation of the sample to exceed the mean.

Key Takeaways

- Probability distribution functions (PDFs) discussed in the chapter are used in several transportation engineering modeling applications. For example, arrival patterns at a roadway location critical to queue analysis follow a Poisson’s distribution while the time interval between arrivals follows a negative binomial distribution.
- Negative binomial distribution and its variants are also used to model collision frequency on road segments and intersections. These models, known as Safety Performance Functions, are used to estimate expected crash frequency on roadway entities.
- Data from spot speed studies are used to ensure that the roadway design context aligns with the

posted speed limit. The speeds on the roadway segments with no congestion follow a normal distribution, and the data are used to estimate the mean, median, and 85th percentile of the normal distribution.

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CHAPTER 9: DATA ANALYSIS - HYPOTHESIS TESTING, ESTIMATING SAMPLE SIZE, AND MODELING

This chapter provides the foundational concepts and tools for analyzing data commonly seen in the transportation profession. The concepts include hypothesis testing, assessing the adequacy of the sample sizes, and estimating the least square model fit for the data. These applications are useful in collecting and analyzing travel speed data, conducting before-after comparisons, and studying the association between variables, e.g., travel speed and congestion as measured by traffic density on the road.

Learning Objectives

- Estimate the required sample size for testing
- Use specific significance tests including, z-test, t-test (one and two samples), chi-squared test
- Compute corresponding p-value for the tests
- Compute and interpret simple linear regression between two variables.
- Estimate a least-squares fit of data.
- Find confidence intervals for parameter estimates.
- Use of spreadsheet tools (e.g., MS Excel) and basic programming (e.g., R or SPSS) to calculate complex and repetitive mathematical problems similar to earthwork estimates (cut, fill, area, etc.), trip generation and distribution, and linear optimization.
- Use of spreadsheet tools (e.g., MS Excel) and basic programming (e.g., R or SPSS) to create relevant graphs and charts from data points.
- Identify topics in the introductory transportation engineering courses that build on the concepts discussed in this chapter

CENTRAL LIMIT THEOREM

In this section, you will learn about the the central limit theorem by reading each description along with watching the videos. Also, short problems to check your understanding are included.

CENTRAL LIMIT THEOREM



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THE CENTRAL LIMIT THEOREM FOR SAMPLE MEANS

The sampling distribution is a theoretical distribution. It is created by taking many samples of size n from a population. Each sample mean is then treated like a single observation of this new distribution, the sampling distribution. The genius of thinking this way is that it recognizes that when we sample, we are creating an observation and that observation must come from some particular distribution. The Central Limit Theorem answers the question: from what distribution does a sample mean come? If this is discovered, then we can treat a sample mean just like any other observation and calculate probabilities about what values it might take on. We have effectively moved from the world of statistics where we know only what we have from the sample to the world of probability where we know the distribution from which the same mean came and the parameters of that distribution.

The reasons that one samples a population are obvious. The time and expense of checking every invoice to determine its validity or every shipment to see if it contains all the items may well exceed the cost of errors in billing or shipping. For some products, sampling would require destroying them, called destructive sampling. One such example is measuring the ability of a metal to withstand saltwater corrosion for parts on ocean going vessels.

Sampling thus raises an important question; just which sample was drawn. Even if the sample were randomly drawn, there are theoretically an almost infinite number of samples. With just 100 items, there are more than 75 million unique samples of size five that can be drawn. If six are in the sample, the number of possible samples increases to just more than one billion. Of the 75 million possible samples, then, which one did you get? If there is variation in the items to be sampled, there will be variation in the samples. One could draw an “unlucky” sample and make very wrong conclusions concerning the population. This recognition that any sample we draw is really only one from a distribution of samples provides us with what is probably the single most important theorem in statistics: the Central Limit Theorem. Without the Central Limit Theorem, it would be impossible to proceed to inferential statistics from simple probability theory. In its most basic form, the Central Limit Theorem states that regardless of the underlying probability density function of the population data, the theoretical distribution of the means of samples from the population will be normally distributed. In essence, this says that the mean of a sample should be treated like an observation drawn from a normal distribution. The Central Limit Theorem only holds if the sample size is “large enough” which has been shown to be only 30 observations or more.

Figure 1 graphically displays this very important proposition.

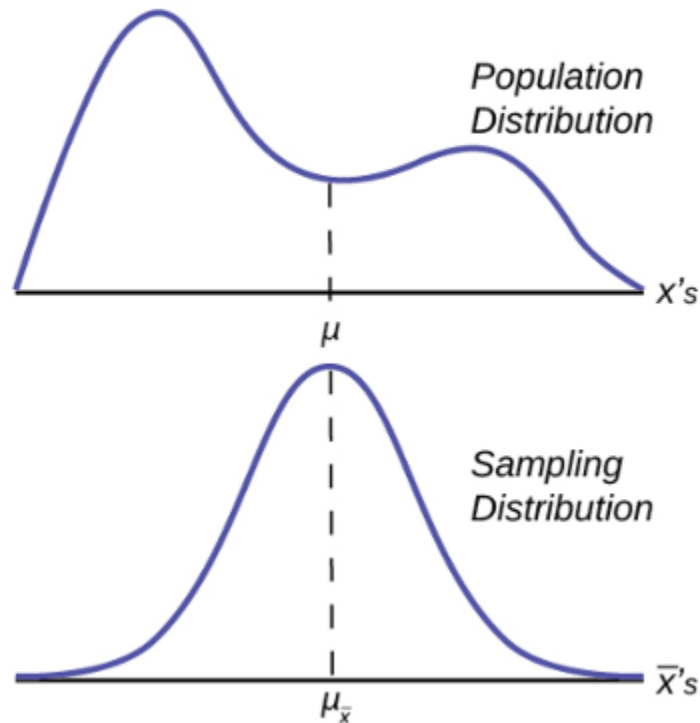


Figure 1

Notice that the horizontal axis in the top panel is labeled X . These are the individual observations of the population. This is the unknown distribution of the population values. The graph is purposefully drawn all squiggly to show that it does not matter just how odd ball it really is. Remember, we will never know what this distribution looks like, or its mean or standard deviation for that matter.

The horizontal axis in the bottom panel is labeled \bar{X} 's. This is the theoretical distribution called the sampling distribution of the means. Each observation on this distribution is a sample mean. All these sample means were calculated from individual samples with the same sample size. The theoretical sampling distribution contains all of the sample mean values from all the possible samples that could have been taken from the population. Of course, no one would ever actually take all of these samples, but if they did this is how they would look. And the Central Limit Theorem says that they will be normally distributed.

The Central Limit Theorem goes even further and tells us the mean and standard deviation of this theoretical distribution.

Table 1

Parameter	Population distribution	Sample	Sampling distribution of \bar{X} 's
Mean	μ	\bar{X}	$\mu_{\bar{x}}$ and $E(\mu_{\bar{x}}) = \mu$
Standard deviation	σ	s	$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$

The practical significant of The Central Limit Theorem is that now we can compute

probabilities for drawing a sample mean, \bar{X} , in just the same way as we did for drawing specific observations, X 's, when we knew the population mean and standard deviation and that the population data were normally distributed. The standardizing formula has to be amended to recognize that the mean and standard deviation of the sampling distribution, sometimes, called the standard error of the mean, are different from those of the population distribution, but otherwise nothing has changed. The new standardizing formula is

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{X} - \mu}{\sigma_{\bar{X}}}$$

Notice that $\mu_{\bar{X}}$ in the first formula has been changed to simply μ in the second version. The reason is that mathematically it can be shown that the expected value of $\mu_{\bar{X}}$ is equal to μ .

SAMPLING DISTRIBUTION OF THE SAMPLE MEAN



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Sampling Distribution of the Sample Mean (Part 2)



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Sampling Distributions: Sampling Distribution of the Mean



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USING THE CENTRAL LIMIT THEOREM

Law of Large Numbers

The **law of large numbers** says that if you take samples of larger and larger size from any population, then the mean of the sampling distribution, $\mu_{\bar{X}}$ tends to get close and close to the true population mean, μ . From the Central Limit Theorem, we know that as n gets larger and larger, the sample means follow a normal distribution. The larger n gets, the smaller the

standard deviation of the sampling distribution gets. (Remember that the standard deviation for sampling distribution of \bar{X} is $\frac{\sigma}{\sqrt{n}}$. This means that the sample mean \bar{x} must be closer to the population mean μ as n increases. We can say that μ is the value that the sample means approach as n gets larger. The Central Limit Theorem illustrates the law of large numbers.

Indeed, there are two critical issues that flow from the Central Limit Theorem and the application of the Law of Large numbers to it. These are listed below.

1. The probability density function of the sampling distribution of means is normally distributed regardless of the underlying distribution of the population observations and
2. Standard deviation of the sampling distribution decreases as the size of the samples that were used to calculate the means for the sampling distribution increases.

Taking these in order. It would seem counterintuitive that the population may have any distribution and the distribution of means coming from it would be normally distributed. With the use of computers, experiments can be simulated that show the process by which the sampling distribution changes as the sample size is increased. These simulations show visually the results of the mathematical proof of the Central Limit Theorem.

Figure 2 shows a sampling distribution. The mean has been marked on the horizontal axis of the \bar{x} 's and the standard deviation has been written to the right above the distribution. Notice that the standard deviation of the sampling distribution is the original standard deviation of the population, divided by the sample size. We have already seen that as the sample size increases the sampling distribution becomes closer and closer to the normal distribution. As this happens, the standard deviation of the sampling distribution changes in another way; the standard deviation decreases as n increases. At very large n , the standard deviation of the sampling distribution becomes very small and at infinity it collapses on top of the population mean. This is what it means that the expected value of $\mu_{\bar{x}}$ is the population mean, μ .

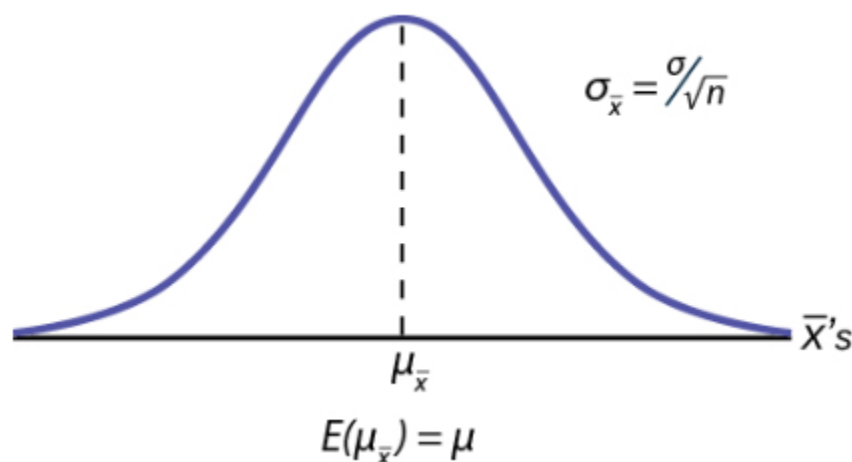


Figure 2

At non-extreme values of n , this relationship between the standard deviation of the sampling

distribution and the sample size plays a very important part in our ability to estimate the parameters in which we are interested.

Figure 3 shows three sampling distributions. The only change that was made is the sample size that was used to get the sample means for each distribution. As the sample size increases, n goes from 10 to 30 to 50, the standard deviations of the respective sampling distributions decrease because the sample size is in the denominator of the standard deviations of the sampling distributions.

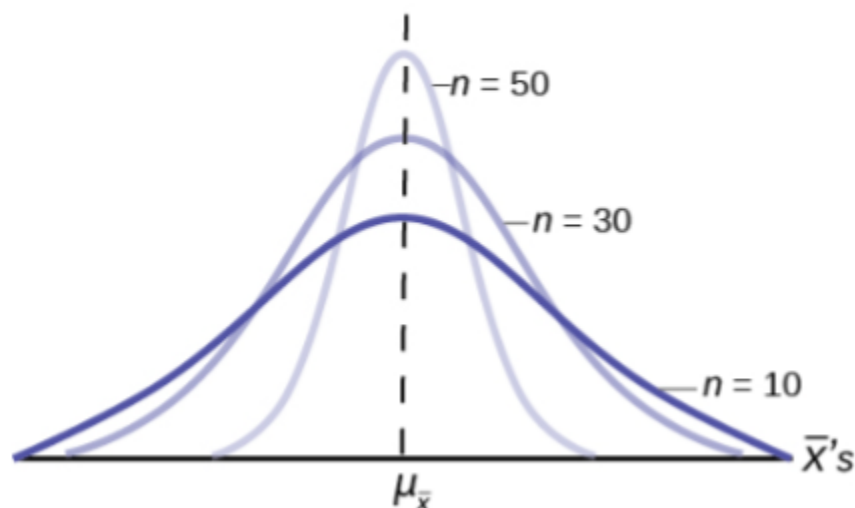


Figure 3

The implications for this are very important. Figure 4 shows the effect of the sample size on the confidence we will have in our estimates. These are two sampling distributions from the same population. One sampling distribution was created with samples of size 10 and the other with samples of size 50. All other things constant, the sampling distribution with sample size 50 has a smaller standard deviation that causes the graph to be higher and narrower. The important effect of this is that for the same probability of one standard deviation from the mean, this distribution covers much less of a range of possible values than the other distribution. One standard deviation is marked on the \bar{X} axis for each distribution. This is shown by the two arrows that are plus or minus one standard deviation for each distribution. If the probability that the true mean is one standard deviation away from the mean, then for the sampling distribution with the smaller sample size, the possible range of values is much greater. A simple question is, would you rather have a sample mean from the narrow, tight distribution, or the flat, wide distribution as the estimate of the population mean? Your answer tells us why people intuitively will always choose data from a large sample rather than a small sample. The sample mean they are getting is coming from a more compact distribution.

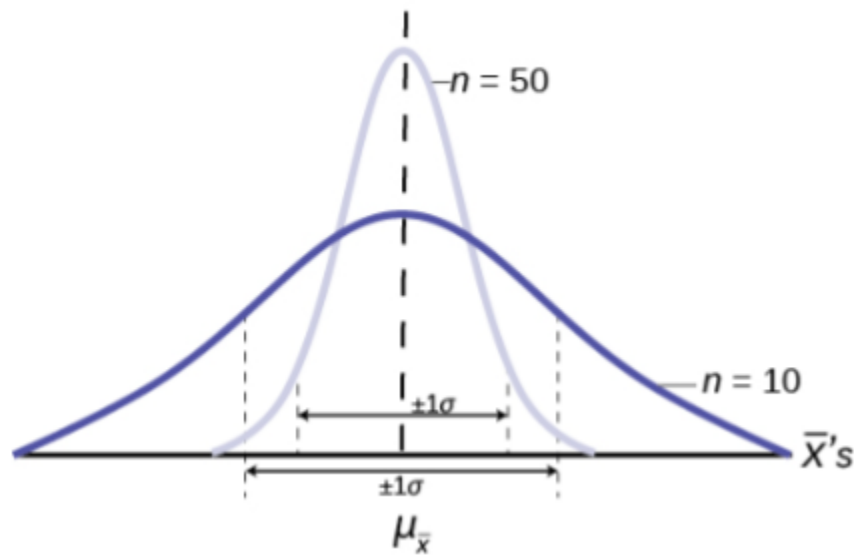


Figure 4

THE CENTRAL LIMIT THEOREM FOR PROPORTIONS

The Central Limit Theorem tells us that the point estimate for the sample mean, \bar{x} , comes from a normal distribution of \bar{x}' s. This theoretical distribution is called the sampling distribution of \bar{x}' s. We now investigate the sampling distribution for another important parameter we wish to estimate, p from the binomial probability density function.

If the random variable is discrete, such as for categorical data, then the parameter we wish to estimate is the population proportion. This is, of course, the probability of drawing a success in any one random draw. Unlike the case just discussed for a continuous random variable where we did not know the population distribution of X 's, here we actually know the underlying probability density function for these data; it is the binomial. The random variable is X = the number of successes and the parameter we wish to know is p , the probability of drawing a success which is of course the proportion of successes in the population. The question at issue is: from what distribution was the sample proportion, $p' = \frac{x}{n}$ drawn? The sample size is n and X is the number of successes found in that sample. This is a parallel question that was just answered by the Central Limit Theorem: from what distribution was the sample mean, \bar{x} drawn? We saw that once we knew that the distribution was the Normal distribution then we were able to create confidence intervals for the population parameter, μ . We will also use this same information to test hypotheses about the population mean later. We wish now to be able to develop confidence intervals for the population parameter " p " from the binomial probability density function.

In order to find the distribution from which sample proportions come we need to develop the sampling distribution of sample proportions just as we did for sample means. So again, imagine that we randomly sample say 50 people and ask them if they support the new school bond issue. From this we find a sample proportion, p' , and graph it on the axis of p 's. We do this again and again etc., etc. until we have the theoretical distribution of p 's. Some sample proportions will show high favorability toward the bond issue and others will show low favorability because

random sampling will reflect the variation of views within the population. What we have done can be seen in Figure 5. The top panel is the population distributions of probabilities for each possible value of the random variable X . While we do not know what the specific distribution looks like because we do not know p , the population parameter, we do know that it must look something like this. In reality, we do not know either the mean or the standard deviation of this population distribution, the same difficulty we faced when analyzing the X 's previously.

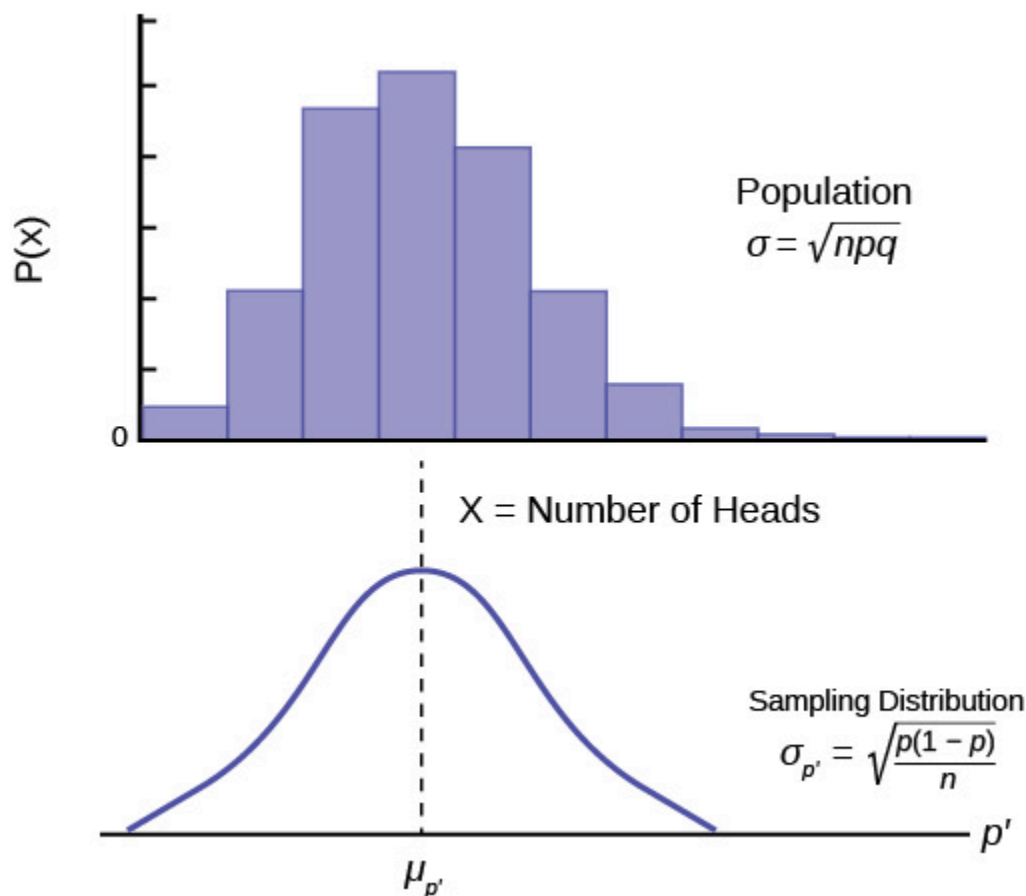


Figure 5

Figure 5 places the mean on the distribution of population probabilities as $\mu = np$ but of course we do not actually know the population mean because we do not know the population probability of success, p . Below the distribution of the population values is the sampling distribution of p 's. Again, the Central Limit Theorem tells us that this distribution is normally distributed just like the case of the sampling distribution for \bar{x} 's. This sampling distribution also has a mean, the mean of the p 's, and a standard deviation, $\sigma_{p'}$.

Importantly, in the case of the analysis of the distribution of sample means, the Central Limit Theorem told us the expected value of the mean of the sample means in the sampling distribution, and the standard deviation of the sampling distribution. Again, the Central Limit Theorem provides this information for the sampling distribution for proportions. The answers are:

1. The expected value of the mean of sampling distribution of sample proportions, $\mu_{p'}$, is

the population proportion, p .

2. The standard deviation of the sampling distribution of sample proportions, $\sigma_{p'}$, is the population standard deviation divided by the square root of the sample size, n .

Both these conclusions are the same as we found for the sampling distribution for sample means. However, in this case, because mean and standard deviation of the binomial distribution both rely upon p , the formula for the standard deviation of the sampling distribution requires algebraic manipulation to be useful. The standard deviation of the sampling distribution for proportions is thus:

$$\sigma_{p'} = \sqrt{\frac{p(1-p)}{n}}$$

Table 2

Parameter	Population distribution	Sample	Sampling distribution of p 's
Mean	$\mu = np$	$p' = \frac{x}{n}$	p' and $E(p') = p$
Standard deviation	$\sigma = \sqrt{npq}$		$\sigma_{p'} = \sqrt{\frac{p(1-p)}{n}}$

Table 2 summarizes these results and shows the relationship between the population, sample, and sampling distribution.

Reviewing the formula for the standard deviation of the sampling distribution for proportions we see that as n increases the standard deviation decreases. This is the same observation we made for the standard deviation for the sampling distribution for means. Again, as the sample size increases, the point estimate for either μ or p is found to come from a distribution with a narrower and narrower distribution. We concluded that with a given level of probability, the range from which the point estimate comes is smaller as the sample size, n , increases.

FIND CONFIDENCE INTERVALS FOR PARAMETER ESTIMATES

In this section, you will learn how to find and estimate confidence intervals by reading each description along with watching the videos included. Also, short problems to check your understanding are included.

CONFIDENCE INTERVALS & ESTIMATION: POINT ESTIMATES EXPLAINED



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INTRODUCTION TO CONFIDENCE INTERVALS

Suppose you were trying to determine the mean rent of a two-bedroom apartment in your

town. You might look in the classified section of the newspaper, write down several rents listed, and average them together. You would have obtained a point estimate of the true mean. If you are trying to determine the percentage of times you make a basket when shooting a basketball, you might count the number of shots you make and divide that by the number of shots you attempted. In this case, you would have obtained a point estimate for the true proportion the parameter p in the binomial probability density function.

We use sample data to make generalizations about an unknown population. This part of statistics is called inferential statistics. The sample data help us to make an estimate of a population parameter. We realize that the point estimate is most likely not the exact value of the population parameter, but close to it. After calculating point estimates, we construct interval estimates, called confidence intervals. What statistics provides us beyond a simple average, or point estimate, is an estimate to which we can attach a probability of accuracy, what we will call a confidence level. We make inferences with a known level of probability.

If you worked in the marketing department of an entertainment company, you might be interested in the mean number of songs a consumer downloads a month from iTunes. If so, you could conduct a survey and calculate the sample mean, \bar{x} , and the sample standard deviation, s . You would use \bar{x} to estimate the population mean and s to estimate the population standard deviation. The same mean, \bar{x} , is the point estimate for the population mean, μ . The sample standard deviation, s , is the point estimate for the population standard deviation, σ .

\bar{x} and s are each called a statistic.

A **confidence interval** is another type of estimate but, instead of being just one number, it is an interval of numbers. The interval of numbers is a range of values calculated from a given set of sample data. The confidence interval is likely to include the unknown population parameter.

Suppose for the iTunes examples, we do not know the population mean μ , but we do know that the population standard deviation is $\sigma = 1$ and our sample size is 100. Then, by the Central Limit Theorem, the standard deviation of the sampling distribution of the sample means is $\frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{100}} = 0.1$.

The **Empirical Rule**, which applies to the normal distribution, says that in approximately 95% of the samples, the same mean, \bar{x} , will be within two standard deviations of the population mean μ . For our iTunes example, two standard deviations is $(2)(0.1) = 0.2$. The sample mean \bar{x} is likely to be within 0.2 units of μ .

Because \bar{x} is within 0.2 units of μ , which is unknown, then μ is likely to be within 0.2 units of \bar{x} with 95% probability. The population mean μ is contained in an interval whose lower number is calculated by taking the sample mean and subtracting two standard deviations $(2)(0.1)$ and whose upper number is calculated by taking the sample mean and adding two standard deviations. In other words, μ is between $\bar{x} - 0.2$ and $\bar{x} + 0.2$ in 95% of all the samples.

For the iTunes example, suppose that a sample produced a sample mean $\bar{x} = 2$. Then with 95% probability the unknown population mean μ is between

$$\bar{x} - 0.2 = 2 - 0.2 = 1.8 \text{ and } \bar{x} + 0.2 = 2 + 0.2 = 2.2$$

We say that we are **95% confident** that the unknown population mean number of songs downloaded from iTunes per month is between 1.8 and 2.2. **The 95% confidence interval is (1.8, 2.2).** Please note that we talked in terms of 95% confidence using the empirical rule. The empirical rule for two standard deviations is only approximately 95% of the probability under the normal distribution. To be precise, two standard deviations under a normal distribution is actually 95.44% of the probability. To calculate the exact 95% confidence level, we would use 1.96 standard deviations.

The 95% confidence interval implies two possibilities. Either the interval (1.8, 2.2) contains the true mean μ , or our sample produce an \bar{x} that is not within 0.2 units of the true mean μ . The first possibility happens for 95% of well-chosen samples. It is important to remember that the second possibility happens for 5% of samples, even though correct procedures are followed.

Remember that a confidence interval is created for an unknown population parameter like the population mean, μ .

For the confidence interval for a mean the formula would be:

$$\mu = \bar{X} \pm Z_{\alpha}\sigma/\sqrt{n}$$

Or written another way as:

$$\bar{X} - Z_{\alpha}\sigma/\sqrt{n} \leq \mu \leq \bar{X} + Z_{\alpha}\sigma/\sqrt{n}$$

Where \bar{X} is the sample mean. Z_{α} is determined by the level of confidence desired by the analyst, and σ/\sqrt{n} is the standard deviation of the sampling distribution for means given to us by the Central Limit Theorem.

A CONFIDENCE INTERVAL FOR A POPULATION STANDARD DEVIATION, KNOWN OR LARGE SAMPLE SIZE

A confidence interval for a population mean, when the population standard deviation is known based on the conclusion of the Central Limit Theorem that the sampling distribution of the sample means follow an approximately normal distribution.

Calculating the Confidence Interval

Consider the standardizing formula for the sampling distribution developed in the discussion of the Central Limit Theorem:

$$Z_1 = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Notice that μ is substituted for $\mu_{\bar{x}}$ because we know that the expected value of $\mu_{\bar{x}}$ is from the Central Limit Theorem and $\sigma_{\bar{x}}$ is replaced with σ/\sqrt{n} , also from the Central Limit Theorem.

In this formula we know \bar{X} , $\sigma_{\bar{x}}$ and n , the sample size. (In actuality we do not know the population standard deviation, but we do have a point estimate for it, s , from the sample we took. More on this later.) What we do not know is μ of Z_{-1} . We can solve for either one of these in terms of the other. Solving for μ in terms of Z_{-1} gives:

$$\mu = \bar{X} \pm Z_1 \sigma / \sqrt{n}$$

Remembering that the Central Limit Theorem tells us that the distribution of the \bar{X}' s, the sampling distribution for means, is normal, and that the normal distribution is symmetrical, we can rearrange terms thus:

$$\bar{X} - Z_\alpha(\sigma/\sqrt{n}) \leq \mu \leq \bar{X} + Z_\alpha(\sigma/\sqrt{n})$$

This is the formula for a confidence interval for the mean of a population.

Notice that Z_α has been substituted for Z_1 in this equation. This is where the statistician must make a choice. The analyst must decide the level of confidence they wish to impose on the confidence interval. α is the probability that the interval will not contain the true population mean. The confidence level is defined as $(1 - \alpha)$. Z_α is the number of standard deviations \bar{X} lies from the mean with a certain probability. If we chose, $Z_\alpha = 1.96$ we are asking for the 95% confidence interval because we are setting the probability that the true mean lies within the range at 0.95. If we set Z_α at 1.64 we are asking for the 90% confidence interval because we have set the probability at 0.90. These numbers can be verified by consulting the Standard Normal table. Divide either 0.95 or 0.90 in half and find that probability inside the body of the table. Then read on the top and left margins the number of standard deviations it takes to get this level of probability.

Table 3

Confidence Level	
0.80	1.28
0.90	1.645
0.95	1.96
0.99	2.58

In reality, we can set whatever level of confidence we desire simply by changing the Z_α value in the formula. It is the analyst's choice. Common convention in Economics and most social sciences sets confidence intervals at either 90, 95, or 99 percent levels. Levels less than 90% are considered of little value. The level of confidence of a particular interval estimate is called $(1 - \alpha)$.

Let us say we know that the actual population mean number of iTunes downloads is 2.1. The true population mean falls within the range of the 95% confidence interval. There is absolutely nothing to guarantee that this will happen. **Further, if the true mean falls outside of the interval, we will never know it. We must always remember that we will never ever know the true mean.** Statistics simply allows us, with a given level of probability (confidence), to say that the true mean is within the range calculated.

Changing the Confidence Level or Sample Size

Here again is the formula for a confidence interval for an unknown population mean assuming we know the population standard deviation:

$$\bar{X} - Z_{\alpha}(\sigma/\sqrt{n}) \leq \mu \leq \bar{X} + Z_{\alpha}(\sigma/\sqrt{n})$$

It is clear that the confidence interval is driven by two things, the chosen level of confidence, Z_{α} , and the standard deviation of the sampling distribution. The standard deviation of the sampling distribution is further affected by two things, the standard deviation of the population and sample size we chose for our data. Here we wish to examine the effects of each of the choices we have made on the calculated confidence interval, the confidence level, and the sample size.

For a moment we should ask just what we desire in a confidence interval. Our goal was to estimate the population mean from a sample. We have forsaken the hope that we will ever find the true population mean, and population standard deviation for that matter, for any case except where we have an extremely small population and the cost of gathering the data of interest is very small. In all other cases we must rely on samples. With the Central Limit Theorem, we have the tools to provide a meaningful confidence interval with a given level of confidence, meaning a known probability of being wrong. By meaningful confidence interval we mean one that is useful. Imagine that you are asked for a confidence interval for the ages of your classmates. You have taken a sample and find a mean of 19.8 years. You wish to be very confident, so you report an interval between 9.8 years and 29.8 years. This interval would certainly contain the true population mean and have a very high confidence level. However, it hardly qualifies as meaningful. The very best confidence interval is narrow while having high confidence. There is a natural tension between these two goals. The higher the level of confidence the wider the confidence interval as the case of the students' ages above. We can see this tension in the equation for the confidence interval.

$$\mu = \bar{x} \pm Z_{\alpha} \left(\frac{\sigma}{\sqrt{n}} \right)$$

The confidence interval will increase in width as Z_{α} increases, Z_{α} increases as the level of confidence increases. There is a tradeoff between the level of confidence and the width of the interval. Now let us look at the formula again and we see that the sample size also plays an important role in the width of the confidence interval. The sample size, n , shows up in the denominator of the standard deviation of the sampling distribution. As the sample size increases, the standard deviation of the sampling distribution decreases and thus the width of the confidence interval, while holding constant the level of confidence. Again, we see the importance of having large samples for our analysis although we then face a second constraint, the cost of gathering data.

Calculating the Confidence Interval: An Alternative Approach

Another way to approach confidence intervals is through the use of something called the Error Bound (margin of error). The Error Bound gets its name from the recognition that it provides the boundary of the interval derived from the standard error of the sampling distribution. In the

equations above it is seen that the interval is simply the estimated mean, sample mean, plus or minus something. That something is the Error Bound and is driven by the probability we desire to maintain in our estimate, Z_α , times the standard deviation of the sampling distribution. The Error Bound for a mean is given the name, **Error Bound Mean**, or EBM (or margin of error, M.O.E.).

To construct a confidence interval for a single unknown population, mean μ , **where the population standard deviation is known**, we need \bar{x} as an estimate for μ and we need the margin of error. Here, the margin of error (EBM) is called the error bound for a population mean. The sample mean \bar{x} is the **point estimate** of the unknown population mean μ .

The confidence interval estimate will have the form:

(Point estimate – error bound, point estimate + error bound) or, in symbols,

$$(\bar{x} - EBM, \bar{x} + EBM).$$

The mathematical formula for this confidence interval is:

$$\bar{X} - Z_\alpha(\sigma/\sqrt{n}) \leq \mu \leq \bar{X} + Z_\alpha(\sigma/\sqrt{n})$$

The margin of error (EBM) depends on the **confidence level** (abbreviated **CL**). The confidence level is often considered the probability that the calculated confidence interval estimate will contain the true population parameter. However, it is more accurate to state that the confidence level is the percent of confidence intervals that contain the true population parameter when repeated samples are taken. Most often, it is the choice of the person constructing the confidence interval to choose a confidence level of 90% or higher because that person wants to be reasonably certain of his or her conclusions.

There is another probability called alpha (α). α is related to the confidence level, **CL**. α is the probability that the interval does not contain the unknown population parameter. Mathematically, $1 - \alpha = CL$.

A confidence interval for a population mean with a known standard deviation is based on the fact that the sampling distribution of the sample means follow an approximately normal distribution. Suppose that our sample has a mean of $\bar{x} = 10$, and we have constructed the 90% confidence interval (5, 15) where EMB = 5.

To get a 90% confidence interval, we must include the central 90% of the probability of the normal distribution. If we include the central 90%, we leave out a total of $\alpha = 10\%$ in both tails, or 5% in each tail, of the normal distribution.

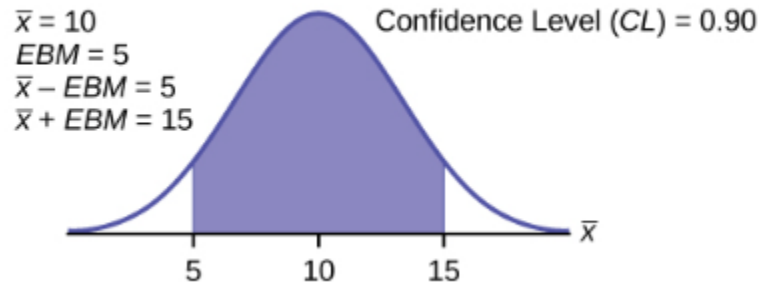


Figure 6

To capture the central 90%, we must go out 1.645 standard deviations on either side of the calculated sample mean. The value 1.645 is the z-score from a standard normal probability distribution that puts an area of 0.90 in the center, an area of 0.05 in the far-left tail, and an area of 0.05 in the far-right tail.

It is important that the standard deviation used must be appropriate for the parameter we are estimating, so in this section we need to use the standard deviation that applies to the sampling distribution for mean which we studied with the Central Limit Theorem and is, $\frac{\sigma}{\sqrt{n}}$.

Calculating the Confidence Interval Using EBM

To construct a confidence interval, estimate for an unknown population mean, we need data from a random sample. The steps to construct and interpret the confidence interval are listed below.

- Calculate the sample mean \bar{x} from the sample data. Remember, in this section we know the population standard deviation σ .
- Find the z-score from the standard normal table that corresponds to the confidence level desired.
- Calculate the error bound EBM.
- Construct the confidence interval.
- Write a sentence that interprets the estimate in the context of the situation in the problem.

Finding the z-score for the Stated Confidence Level

When we know the population standard deviation σ , we use a standard normal distribution to calculate the error bound EBM and construct the confidence interval. We need to find the value of z that puts an area equal to the confidence level (in decimal form) in the middle of the standard normal distribution $Z \sim N(0, 1)$.

The confidence level, CL , is the area in the middle of the standard normal distribution. $CL = 1 - \alpha$, so α is the area that is split equally between the two tails. Each of the tails contains an area equal to $\frac{\alpha}{2}$.

The z-score that has an area to the right of $\frac{\alpha}{2}$ is denoted by $Z_{\frac{\alpha}{2}}$

For example, when $CL = 0.95$, $\alpha = 0.05$ and $\frac{\alpha}{2} = 0.025$; we write $Z_{\frac{\alpha}{2}} = Z_{0.025}$

The area to the right of $Z_{0.025}$ is 0.025 and the area to the left of $Z_{0.025}$ is $1 - 0.025 = 0.975$.

$Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96$, using a standard normal probability table. We will see later that we can use a different probability table, the Student's t-distribution, for finding the number of standard deviations of commonly used levels of confidence.

Calculating the Error Bound (EBM)

The error bound formula for an unknown population mean μ when the population standard deviation σ is known as

$$EBM = \left(Z_{\frac{\alpha}{2}} \right) \left(\frac{\sigma}{\sqrt{n}} \right)$$

Constructing the Confidence Interval

The confidence interval estimate has the format $(\bar{x} - EBM, \bar{x} + EBM)$ or the formula:

$$\bar{X} - Z_{\alpha} \left(\frac{\sigma}{\sqrt{n}} \right) \leq \mu \leq \bar{X} + Z_{\alpha} \left(\frac{\sigma}{\sqrt{n}} \right)$$

The graph gives a picture of the entire situation.

$$CL + \frac{\alpha}{2} + \frac{\alpha}{2} = CL + \alpha = 1$$

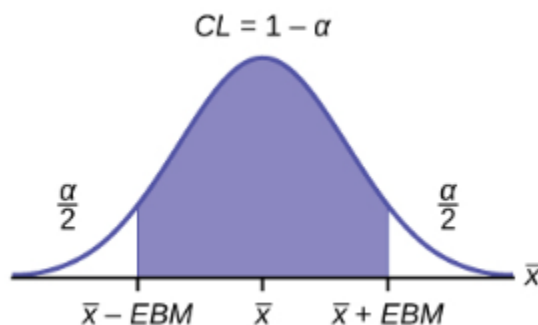


Figure 7

Confidence Interval for Mean: 1 Sample Z Test (Using Formula)



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Check Your Understanding: Confidence Interval for Mean



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A CONFIDENCE INTERVAL FOR A POPULATION STANDARD DEVIATION UNKNOWN, SMALL SAMPLE CASE

In practice, we rarely know the population standard deviation. In the past, when the sample size was large, this did not present a problem to statisticians. They used the sample standard deviation s as an estimate for σ and proceeded as before to calculate a confidence interval with close enough results. Statisticians ran into problems when the sample size was small. A small sample size caused inaccuracies in the confidence interval.

William S. Goset (1876-1937) of the Guinness brewery in Dublin, Ireland ran into this problem. His experiments with hops and barley produced very few samples. Just replacing σ with s did not produce accurate results when he tried to calculate a confidence interval. He realized that he could not use a normal distribution for the calculation; he found that the actual distribution depends on the sample size. This problem led him to “discover” what is called the **Student’s t-distribution**. The name comes from the fact that Goset wrote under the pen name “A Student.”

Up until the mid-1970s, some statisticians used **the normal distribution** approximation for large sample sizes and used the Student’s t-distribution only for sample sizes of at most 30 observations.

If you draw a simple random sample of size n from a population with mean μ and unknown population standard deviation σ and calculate the t-score $t = \frac{\bar{x} - \mu}{\left(\frac{s}{\sqrt{n}}\right)}$, then the t-scores follow a **Student’s t-distribution with $n - 1$ degrees of freedom**. The t-score has the same interpretation as the z-score. It measures how far in standard deviation units \bar{x} is from its mean μ . For each sample size n , there is a different Student’s t-distribution.

The **degrees of freedom**, $n - 1$, come from the calculation of the sample standard deviation s . Remember when we first calculated a sample standard deviation, we divided the sum of the squared deviations by $n - 1$, but we used n deviations ($x - \bar{x}$ values) to calculate s . Because the sum of the deviations is zero, we can find the last deviation once we know the other $n - 1$ deviations. The other $n - 1$ deviations can change or vary freely. We call the number $n - 1$ the degrees of freedom (df) in recognition that one is lost in the calculations. The effect of losing a degree of freedom is that the t-value increases, and the confidence interval increases in width.

Properties of the Student's t-distribution

- The graph for the Student's t-distribution is similar to the standard normal curve and at infinite degrees of freedom it is the normal distribution. You can confirm this by reading the bottom line at infinite degrees of freedom for a familiar level of confidence, e.g., at column 0.05, 95% level of confidence, we find the t-value of 1.96 at infinite degrees of freedom.
- The mean for the Student's t-distribution is zero and the distribution is symmetric about zero, again like the standard normal distribution.
- The Student's t-distribution has more probability in its tails than the standard normal distribution because the spread of the t-distribution is greater than the spread of the standard normal. So, the graph of the Student's t-distribution will be thicker in the tails and shorter in the center than the graph of the standard normal distribution.
- The exact shape of the Student's t-distribution depends on the degrees of freedom. As the degrees of freedom increases, the graph of Student's t-distribution becomes more like the graph of the standard normal distribution.
- The underlying population of individual observations is assumed to be normally distributed with unknown population mean μ and unknown population standard deviation σ . This assumption comes from the Central Limit Theorem because the individual observations in this case are the \bar{x} 's of the sample distribution. The size of the underlying population is generally not relevant unless it is very small. If it is normal, then the assumption is met and does not need discussion.

A probability table for the Student's t-distribution is used to calculate t-values at various commonly used levels of confidence. The table gives t-scores that correspond to the confidence level (column) and degrees of freedom (row). When using a t-table, note that some tables are formatted to show the confidence level in the column headings, while the column headings in some tables may show only corresponding area in one or both tails. Notice that at the bottom the table will show the t-value for infinite degrees of freedom. Mathematically, as the degrees of freedom increase, the t-distribution approaches the standard normal distribution. You can find familiar Z-values by looking in the relevant alpha column and reading value in the last row.

A Student's t-table ([link](#)) gives t-scores given the degrees of freedom and the right-tailed probability.

The Student's t-distribution has one of the most desirable properties of the normal: it is symmetrical. What the Student's t-distribution does is spread out the horizontal axis, so it takes a larger number of standard deviations to capture the same amount of probability. In reality there are an infinite number of Student's t-distributions, one for each adjustment to the sample size. As the sample size increases, the Student's t-distribution become more and more like the normal distribution. When the sample size reaches 30 the normal distribution is usually substituted for the Student's t because they are so much alike. This relationship between the Student's t-distribution and the normal distribution is shown in Figure 8.

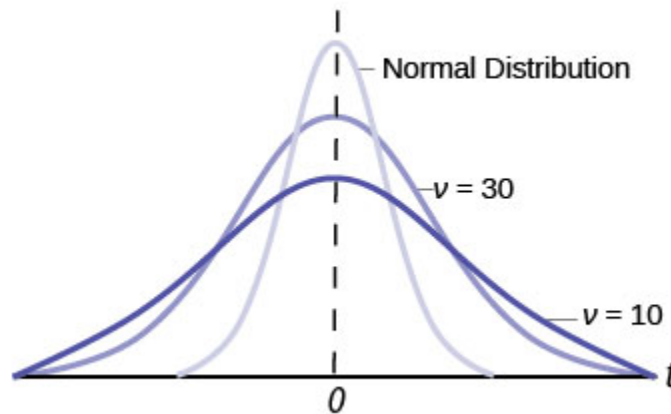


Figure 8

Here is the confidence interval for the mean for cases when the sample size is smaller than 30 and we do not know the population standard deviation, σ :

$$\bar{x} - t_{v,\alpha} \left(\frac{s}{\sqrt{n}} \right) \leq \mu \leq \bar{x} + t_{v,\alpha} \left(\frac{s}{\sqrt{n}} \right)$$

Here the point estimate of the population standard deviation, s has been substituted for the population standard deviation, σ , and $t_{v,\alpha}$ has been substituted for Z_{α} . The Greek letter ν (pronounced nu) is placed in the general formula in recognition that there are many Student t_{ν} distributions, one for each sample size. ν is the symbol for the degrees of freedom of the distribution and depends on the size of the sample. Often df is used to abbreviate degrees of freedom. For this type of problem, the degrees of freedom is $\nu = n - 1$, where n is the sample size. To look up a probability in the Student's t table we have to know the degrees of freedom in the problem.

Confidence Intervals: Using the t Distribution



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Check Your Understanding: Confidence Intervals



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A CONFIDENCE INTERVAL FOR A POPULATION PROPORTION

During an election year, we see articles in the newspaper that state confidence intervals in terms of proportions or percentages. For example, a poll for a particular candidate running for president might show that the candidate has 40% of the vote within three percentage points (if the sample is large enough). Often, election polls are calculated with 95% confidence, so, the pollsters would be 95% confident that the true proportion of voters who favored the candidate would be between 0.37 and 0.43.

The procedure to find the confidence interval for a population proportion is similar to that for the population mean, but the formulas are a bit different although conceptually identical. While the formulas are different, they are based upon the same mathematical foundation given to us by the Central Limit Theorem. Because of this we will see the same basic format using the same three pieces of information: the sample value of the parameter in question, the standard deviation of the relevant sampling distribution, and the number of standard deviations we need to have the confidence in our estimate that we desire.

How do you know you are dealing with a proportion problem? First, the underlying **distribution has a binary random variable and therefore is a binomial distribution**. (There is no mention of a mean or average). If X is a binomial random variable, then $X \sim B(n, p)$ where n is the number of trials and p is the probability of a success. To form a sample proportion, take X , the random variable for the number of successes and divide it by n , the number of trials (or the sample size). The random variable P' (read “P Prime”) is the sample proportion,

$$P' = \frac{X}{n}$$

(Sometimes the random variable is denoted as \hat{P} , read “P hat.”)

p' = the **estimated proportion** of successes or sample proportion of successes (p' is a **point estimate** for p , the true population proportion, and thus q is the probability of a failure in any one trial.)

x = the **number** of successes in the sample

n = the size of the sample

The formula for the confidence interval for a population proportion follows the same format as that for an estimate of a population mean. Remembering the sampling distribution for the proportion, the standard deviation was found to be:

$$\sigma_{p'} = \sqrt{\frac{p(1-p)}{n}}$$

The confidence interval for a population proportion, therefore, becomes:

$$p = p' \pm \left[Z_{\left(\frac{\alpha}{2}\right)} \sqrt{\frac{p'(1-p')}{n}} \right]$$

$Z_{\left(\frac{\alpha}{2}\right)}$ is set according to our desired degree of confidence and $\sqrt{\frac{p'(1-p')}{n}}$ is the standard deviation of the sampling distribution.

The **sample proportions p' and q' are estimates of the unknown population proportions p and q .** The estimated proportions p' and q' are used because p and q are not known.

Remember that as p moves further from 0.5 the binomial distribution becomes less symmetrical. Because we are estimating the binomial with the symmetrical normal distribution the further away from symmetrical the binomial becomes the less confidence we have in the estimate.

This conclusion can be demonstrated through the following analysis. Proportions are based upon the binomial probability distribution. The possible outcomes are binary, either “success” or “failure.” This gives rise to a proportion, meaning the percentage of the outcomes that are “successes.” It was shown that the binomial distribution could be fully understood if we knew only the probability of a success in any one trial, called p . The mean and the standard deviation of the binomial were found to be:

$$\mu = np$$

$$\sigma = \sqrt{npq}$$

It was also shown that the binomial could be estimated by the normal distribution if BOTH np AND nq were greater than 5. From the discussion above, it was found that the standardizing formula for the binomial distribution is:

$$Z = \frac{p' - p}{\sqrt{\left(\frac{pq}{n}\right)}}$$

Which is nothing more than a restatement of the general standardizing formula with appropriate substitutions for μ and σ from the binomial. We can use the standard normal distribution, the reason Z is in the equation, because the normal distribution is the limiting distribution of the binomial. This is another example of the Central Limit Theorem.

We can now manipulate this formula in just the same way we did for finding the confidence intervals for a mean, but to find the confidence interval for the binomial population parameter, p .

$$p' - Z_{\alpha} \sqrt{\frac{p'q'}{n}} \leq p \leq p' + Z_{\alpha} \sqrt{\frac{p'q'}{n}}$$

Where $p' = x/n$, the point estimate of p taken from the sample. Notice that p' has replaced p in the formula. This is because we do not know p , indeed, this is just what we are trying to estimate.

x = number of successes.

n = the number in the sample.

$q' = (1 - p')$, the failure in any trial

Unfortunately, there is no correction factor for cases where the sample size is small so np' and nq' must always be greater than 5 to develop an interval estimate for p .

Also written as:

$$p' - Z_{\alpha} \sqrt{\frac{p'(1-p')}{n}} \leq p \leq p' + Z_{\alpha} \sqrt{\frac{p'(1-p')}{n}}$$

How to Construct a Confidence Interval for Population Proportion



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Check Your Understanding: How to Construct a Confidence Interval for Population Proportion



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ESTIMATE THE REQUIRED SAMPLE SIZE FOR TESTING

In this section, you will learn how to calculate sample size with continuous and binary random

samples. by reading each description along with watching the videos included. Also, short problems to check your understanding are included.

Calculating the Sample Size n : Continuous and Binary Random Variables

Continuous Random Variables

Usually, we have no control over the sample size of a data set. However, if we are able to set the sample size, as in cases where we are taking a survey, it is very helpful to know just how large it should be to provide the most information. Sampling can be very costly in both time and product. Simple telephone surveys will cost approximately \$30.00 each, for example, and some sampling requires the destruction of the product.

If we go back to our standardizing formula for the sampling distribution for means, we can see that it is possible to solve it for n . If we do this, we have $(\bar{X} - \mu)$ in the denominator.

$$n = \frac{Z_{\alpha}^2 \sigma^2}{(\bar{X} - \mu)^2} = \frac{Z_{\alpha}^2 \sigma^2}{e^2}$$

Because we have not taken a sample, yet we do not know any of the variables in the formula except that we can set Z_{α} to the level of confidence we desire just as we did when determining confidence intervals. If we set a predetermined acceptable error, or tolerance, for the difference between \bar{X} and μ , called e in the formula, we are much further in solving for the sample size n . We still do not know the population standard deviation, σ . In practice, a pre-survey is usually done which allows for fine tuning the questionnaire and will give a sample standard deviation that can be used. In other cases, previous information from other surveys may be used for σ in the formula. While crude, this method of determining the sample size may help in reducing cost significantly. It will be the actual data gathered that determines the inferences about the population, so caution in the sample size is appropriate calling for high levels of confidence and small sampling errors.

Binary Random Variables

What was done in cases when looking for the mean of a distribution can also be done when sampling to determine the population parameter p for proportions. Manipulation of the standardizing formula for proportions gives:

$$n = \frac{Z_{\alpha}^2 pq}{e^2}$$

Where $e = (p' - p)$, and is the acceptable sampling error, or tolerance, for this application. This will be measured in percentage points.

In this case the very object of our search is in the formula, p , and of course q because $q = 1 - p$. This result occurs because the binomial distribution is a one parameter distribution. If we know p then we know the mean and the standard deviation. Therefore, p shows up in the standard deviation of the sampling distribution which is where we got this formula. If, in an abundance of caution, we substitute 0.5 for p we will draw the largest required sample size that will provided the level of confidence specified by Z_{α} and the tolerance we have selected. This is

true because of all combinations of two fractions that add to one, the largest multiple is when each is 0.5. Without any other information concerning the population parameter p , this is the common practice. This may result in oversampling, but certainly not under sampling, thus, this is a cautious approach.

There is an interesting trade-off between the level of confidence and the sample size that shows up here when considering the cost of sampling. Table 4 shows the appropriate sample size at different levels of confidence and different level of the acceptable error, or tolerance.

Table 4

Required sample size (90%)	Required sample size (95%)	Tolerance level
1691	2401	2%
752	1067	3%
271	384	5%
68	96	10%

Table 4 is designed to show the maximum sample size required at different levels of confidence given an assumed $p = 0.5$ and $q = 0.5$ as discussed above.

The acceptable error, called tolerance in the table, is measured in plus or minus values from the actual proportion. For example, an acceptable error of 5% means that if the sample proportion was found to be 26 percent, the conclusion would be that the actual population proportion is between 21 and 31 percent with a 90 percent level of confidence if a sample of 271 had been taken. Likewise, if the acceptable error was set at 2%, then the population proportion would be between 24 and 28 percent with a 90 percent level of confidence but would require that the sample size be increased from 271 to 1,691. If we wished a higher level of confidence, we would require a larger sample size. Moving from a 90 percent level of confidence to a 95 percent level at a plus or minus 5% tolerance requires changing the sample size from 271 to 384. A very common sample size often seen reported in political surveys is 384. With the survey results it is frequently stated that the results are good to a plus or minus 5% level of “accuracy”.

Example: Suppose a mobile phone company wants to determine the current percentage of customers aged 50+ who use text messaging on their cell phones. How many customers aged 50+ should the company survey in order to be 90% confident that the estimated (sample) proportion is within three percentage points of the true population proportion of customers aged 50+ who use text messaging on their cell phones.

Solution: From the problem, we know that the acceptable error, e , is 0.03 ($3\% = 0.03$) and $z_{\frac{\alpha}{2}}, z_{0.05} = 1.645$, because the confidence level is 90%. The acceptable error, e , is the difference between the actual population proportion p , and the sample proportion we expect to get from the sample.

However, in order to find n , we need to know the estimated (sample) proportion p' . Remember that $q' = 1 - p'$. But we do not know p' yet. Since we multiply p' and q' together, we make them both equal to 0.5 because $p'q' = (0.5)(0.5) = 0.25$ results in the largest possible product. (Try other products: $(0.6)(0.4) = 0.24$; $(0.3)(0.7) = 0.21$; $(0.2)(0.8) = 0.16$ and so on).

The largest possible product gives us the largest n . This gives us a large enough sample so that we can be 90% confident that we are within three percentage points of the true population proportion. To calculate the sample size n , use the formula and make the substitutions.

$$n = \frac{z^2 p' q'}{e^2} \text{ gives } n = \frac{1.645^2 (0.5)(0.5)}{0.03^2} = 751.7$$

Round the answer to the next higher value. The sample size should be 752 cell phone customers aged 50+ in order to be 90% confident that the estimated (sample) proportion is within three percentage points of the true population proportion of all customers aged 50+ who use text messaging on their cell phones.

ESTIMATION AND CONFIDENCE INTERVALS: CALCULATE SAMPLE SIZE



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CALCULATING SAMPLE SIZE TO PREDICT A POPULATION PROPORTION



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USE SPECIFIC SIGNIFICANCE TESTS INCLUDING, Z-TEST, T-TEST (ONE AND TWO SAMPLES), CHI-SQUARED TEST

In this section, you will learn the fundamentals of hypothesis testing along with hypothesis testing with errors by reading each description along with watching the videos. Also, short problems to check your understanding are included.

HYPOTHESIS TESTING WITH ONE SAMPLE

Statistical testing is part of a much larger process known as the scientific method. The scientific method, briefly, states that only by following a careful and specific process can some assertion be included in the accepted body of knowledge. This process begins with a set of assumptions upon which a theory, sometimes called a model, is built. This theory, if it has any validity, will lead to predictions; what we call hypotheses.

Statistics and statisticians are not necessarily in the business of developing theories, but in the business of testing others' theories. Hypotheses come from these theories based upon an explicit

set of assumptions and sound logic. The hypothesis comes first, before any data are gathered. Data do not create hypotheses; they are used to test them. If we bear this in mind as we study this section, the process of forming and testing hypotheses will make more sense.

One job of a statistician is to make statistical inferences about populations based on samples taken from the population. **Confidence intervals** are one way to estimate a population parameter. Another way to make a statistical inference is to make a decision about the value of a specific parameter. For instance, a car dealer advertises that its new small truck gets 35 miles per gallon, on average. A tutoring service claims that its method of tutoring helps 90% of its students get an A or a B. A company says that women managers in their company earn an average of \$60,000 per year.

A statistician will make a decision about these claims. This process is called “**hypothesis testing**.” A hypothesis test involves collecting data from a sample and evaluating the data. Then, the statistician makes a decision as to whether or not there is sufficient evidence, based upon analyses of the data, to reject the null hypothesis.

Hypothesis Testing: The Fundamentals



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Null and Alternative Hypotheses

The actual test begins by considering two **hypotheses**. They are called the **null hypothesis** and the **alternative hypothesis**. These hypotheses contain opposing viewpoints.

H_0 : **The null hypothesis:** It is a statement of no difference between the variables—they are not related. This can often be considered the status quo and as a result if you cannot accept the null, it requires some action.

H_a : **The alternative hypothesis:** It is a claim about the population that is contradictory to H_0 and what we conclude when we cannot accept H_0 . This is usually what the researcher is trying to prove. The alternative hypothesis is the contender and must win with significant evidence to overthrow the status quo. This concept is sometimes referred to the tyranny of the status quo because as we will see later, to overthrow the null hypothesis takes usually 90 or greater confidence that this is the proper decision.

Since the null and alternative hypotheses are contradictory, you must examine evidence to decide if you have enough evidence to reject the null hypothesis or not. The evidence is in the form of sample data.

After you have determined which hypothesis the sample supports, you make a **decision**. There are two options for a decision. They are “cannot accept H_0 ” if the sample information favors the alternative hypothesis or “do not reject H_0 ” or “decline to reject H_0 ” if the sample information is insufficient to reject the null hypothesis. These conclusions are all based upon a level of probability, a significance level, that is set by the analyst.

Table 5 presents the various hypotheses in the relevant pairs. For example, if the null hypothesis is equal to some value, the alternative has to be not equal to that value.

Table 5

H_0	H_a
Equal (=)	Not equal (\neq)
Greater than or equal to (\geq)	Less than (<)
Less than or equal to (\leq)	More than (>)

Note: As a mathematical convention H_0 always has a symbol with an equal in it. H_a never has a symbol with an equal in it. The choice of symbol depends on the wording of the hypothesis test.

Example 1:

H_0 : No more than 30% of the registered voters in Santa Clara County voted in the primary election. $p \leq .30$

H_a : More than 30% of the registered voters in Santa Clara County voted in the primary election. $p > .30$

Example 2: We want to test whether the mean GPA of students in American colleges is different from 2.0 (out of 4.0). The null and alternative hypotheses are:

$$H_0 : \mu = 2.0$$

$$H_a : \mu \neq 2.0$$

Example 3: We want to test if college students take less than five years to graduate from college, on the average. The null and alternative hypotheses are:

$$H_0 : \mu \geq 5$$

$$H_a : \mu < 5$$

Hypothesis Testing: Setting up the Null and Alternative Hypothesis Statements



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Outcomes and the Type I and Type II Errors

When you perform a hypothesis test, there are four possible outcomes depending on the actual truth (or falseness) of the null hypothesis H_0 and the decision to reject or not. The outcomes are summarized in Table 6:

Table 6		
Statistical Decision	H_0 is actually...	
	True	False
Cannot reject H_0	Correct outcome	Type II error
Cannot accept H_0	Type I error	Correct outcome

The four possible outcomes in the table are:

1. The decision is **cannot reject** H_0 when H_0 is **true (correct decision)**.
2. The decision is **cannot accept** H_0 when H_0 is **true** (incorrect decision known as a **Type I error**). This case is described as “rejecting a good null”. As we will see later, it is this type of error that we will guard against by setting the probability of making such an error. The goal is to NOT take an action that is an error.
3. The decision is **cannot reject** H_0 when, in fact, H_0 is **false** (incorrect decision known as a **Type II error**). This is called “accepting a false null”. In this situation you have allowed the status quo to remain in force when it should be overturned. As we will see, the null hypothesis has the advantage in competition with the alternative.
4. The decision is **cannot accept** H_0 when H_0 is **false (correct decision)**.

Each of the errors occurs with a particular probability. The Greek letters α and β represent the probabilities.

α = probability of a Type I error = **P(Type I error)** = probability of rejecting the null hypothesis when the null hypothesis is true: rejecting a good null.

β = probability of a Type II error = **P(Type II error)** = probability of not rejecting the null hypothesis when the null hypothesis is false. $(1 - \beta)$ is called the **Power of the Test**.

α and β should be as small as possible because they are probabilities of errors.

Statistics allows us to set the probability that we are making a Type I error. The probability of making a Type I error is α . Recall that the confidence intervals in the last section were set by choosing a value called Z_α (or t_α) and the alpha value determined the confidence level of the estimate because it was the probability of the interval failing to capture the true mean (or proportion parameter p). This alpha and that one are the same.

The easiest way to see the relationship between the alpha error and the level of confidence is in Figure 9.

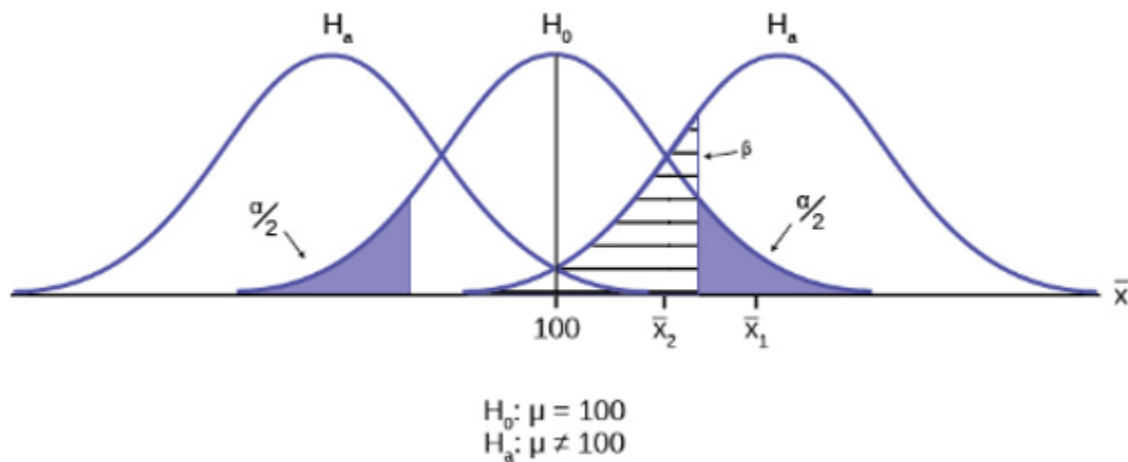


Figure 9

In the center of Figure 9 is a normally distributed sampling distribution marked H_0 . This is a sampling distribution of \bar{X} and by the Central Limit Theorem it is normally distributed. The distribution in the center is marked H_0 and represents the distribution for the null hypotheses $H_0 : \mu = 100$. This is the value that is being tested. The formal statements of the null and alternative hypotheses are listed below the figure.

The distributions on either side of the H_0 distribution represent distributions that would be true if H_0 is false, under the alternative hypothesis listed as H_a . We do not know which is true, and will never know. There are, in fact, an infinite number of distributions from which the data could have been drawn if H_a is true, but only two of them are on Figure 9 representing all of the others.

To test a hypothesis, we take a sample from the population and determine if it could have come from the hypothesized distribution with an acceptable level of significance. This level of significance is the alpha error and is marked on Figure 9 as the shaded areas in each tail of the H_0 distribution. (Each are actually $\alpha/2$ because the distribution is symmetrical, and the alternative hypothesis allows for the possibility for the value to be either greater than or less than the hypothesized value—called a two-tailed test).

If the sample mean marked as \bar{X}_1 is in the tail of the distribution of H_0 , we conclude that the probability that it could have come from the H_0 distribution is less than alpha. We consequently state, “the null hypothesis cannot be accepted with (α) level of significance.” The truth **may** be that this \bar{X}_1 did come from the H_0 distribution, but from out in the tail. If this is so, then we have falsely rejected a true null hypothesis and have made a Type I error. What statistics has done is provide an estimate about what we know, and what we control, and that is the probability of us being wrong, α .

We can also see in Figure 9 that the sample mean could be really from an H_a distribution, but within the boundary set by the alpha level. Such a case is marked as \bar{X}_2 . There is a probability that \bar{X}_2 actually came from H_a but shows up in the range of H_0 between the two tails. This probability is the beta error, the probability of accepting a false null.

Our problem is that we can only set the alpha error because there are an infinite number of alternative distributions from which the mean could have come that are not equal to H_0 . As a result, the statistician places the burden of proof on the alternative hypothesis. That is, we will not reject a null hypothesis unless there is a greater than 90, or 95, or even 99 percent probability that the null is false: the burden of proof lies with the alternative hypothesis. This is why we called this the tyranny of the status quo earlier.

By way of example, the American judicial system begins with the concept that a defendant is “presumed innocent”. This is the status quo and is the null hypothesis. The judge will tell the jury that they cannot find the defendant guilty unless the evidence indicates guilt beyond a “reasonable doubt” which is usually defined in criminal cases as 95% certainty of guilt. If the jury cannot accept the null, innocent, then action will be taken, jail time. The burden of proof always lies with the alternative hypothesis. (In civil cases, the jury needs only to be more than 50% certain of wrongdoing to find culpability, called “a preponderance of the evidence”).

The example above was for a test of a mean, but the same logic applies to tests of hypotheses for all statistical parameters one may wish to test.

Example: Suppose the null hypothesis, H_0 , is Frank’s rock climbing is safe.

Type I error: Frank thinks that his rock-climbing equipment may not be safe when, in fact, it really is safe.

Type II error: Frank thinks that his rock-climbing equipment may be safe when, in fact, it is not safe.

α = **probability** that Frank thinks his rock-climbing equipment may not be safe when, in fact, it really is safe. β = **probability** that Frank thinks his rock-climbing equipment may be safe when, in fact, it is not safe.

Notice that, in this case, the error with the greater consequence is the Type II error. (If Frank thinks his rock-climbing equipment is safe, he will go ahead and use it.)

This is a situation described as “accepting a false null”.

Hypothesis Testing: Type I and Type II Errors



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Check Your Understanding: Hypothesis Testing: Type I and Type II Errors



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Distribution Needed for Hypothesis Testing

Earlier, we discussed sampling distributions. Particular distributions are associated with hypothesis testing. We will perform hypotheses tests of a population mean using a normal distribution or a Student’s t-distribution. (Remember, use a Student’s t-distribution when the population standard deviation is unknown and the sample size is small, where small is considered to be less than 30 observations.) We perform tests of a population proportion using a normal distribution when we can assume that the distribution is normally distributed. We consider this to be true if the sample proportion, p' , times the sample size is greater than 5 and $1 - p'$ times the sample size is also greater than 5. This is the same rule of thumb we used when developing the formula for the confidence interval for a population proportion.

Hypothesis Test for the Mean

Going back to the standardizing formula we can derive the **test statistic** for testing hypotheses concerning means.

$$Z_c = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

The standardizing formula cannot be solved as it is because we do not have μ , the

population mean. However, if we substituted in the hypothesized value of the mean, μ_0 in the formula as above, we can compute a Z value. This is the test statistic for a test of hypothesis for a mean and is presented in Figure 10. We interpret this Z value as the associated probability that a sample with a sample mean of \bar{X} could have come from a distribution with a population mean of H_0 and we call this Z value Z_c for “calculated.” Figure 10 shows this process.

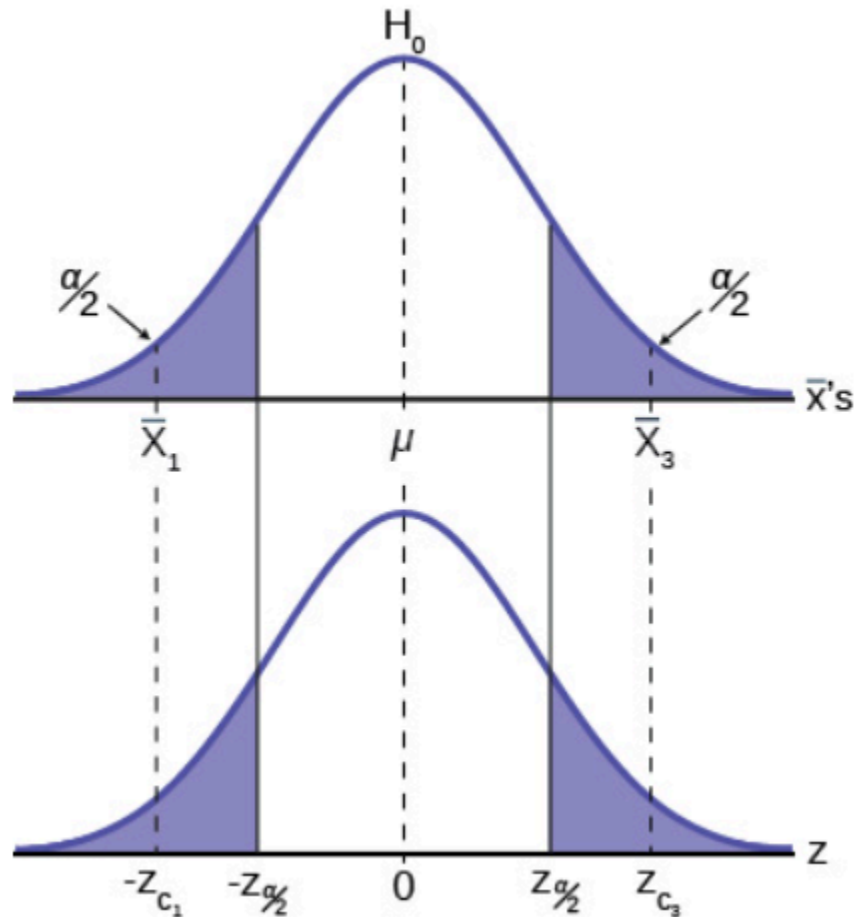


Figure 10

In Figure 10, two of the three possible outcomes are presented. \bar{X}_1 and \bar{X}_3 are in the tails of the hypothesized distribution of H_0 . Notice that the horizontal axis in the top panel is labeled \bar{X}' , \bar{X} 's. This is the same theoretical distribution of \bar{X}' s, the sampling distribution, that the Central Limit Theorem tells us is normally distributed. This is why we can draw it with this shape. The horizontal axis of the bottom panel is labeled Z and is the standard normal distribution. $Z_{\frac{\alpha}{2}}$ and $-Z_{\frac{\alpha}{2}}$, called the **critical values**, are marked on the bottom panel as the Z values associated with the probability the analyst has set as the level of significance in test, (α). The probabilities in the tails of both panels are, therefore, the same.

Notice that for each \bar{X} there is an associated Z_c , called the calculated Z, that comes from solving the equation above. This calculated Z is nothing more than the number of

standard deviations that the **hypothesized** mean is from the sample mean. If the sample mean falls “too many” standard deviations from the hypothesized mean we conclude that the **sample** mean could not have come from the distribution with the hypothesized mean, given our pre-set required level of significance. It **could** have come from H_0 , but it is deemed just too unlikely. In Figure 10, both \bar{X}_1 and \bar{X}_3 are in the tails of the distribution. They are deemed “too far” from the hypothesized value of the mean given the chosen level of alpha. If in fact this sample mean it did come from H_0 , but from in the tail, we have made a Type I error: we have rejected a good null. Our only real comfort is that we know the probability of making such an error, α , and we can control the size of α .

Figure 11 shows the third possibility for the location of the sample mean, \bar{x} . Here the sample mean is within the two critical values. That is, within the probability of $(1 - \alpha)$ and we cannot reject the null hypothesis.

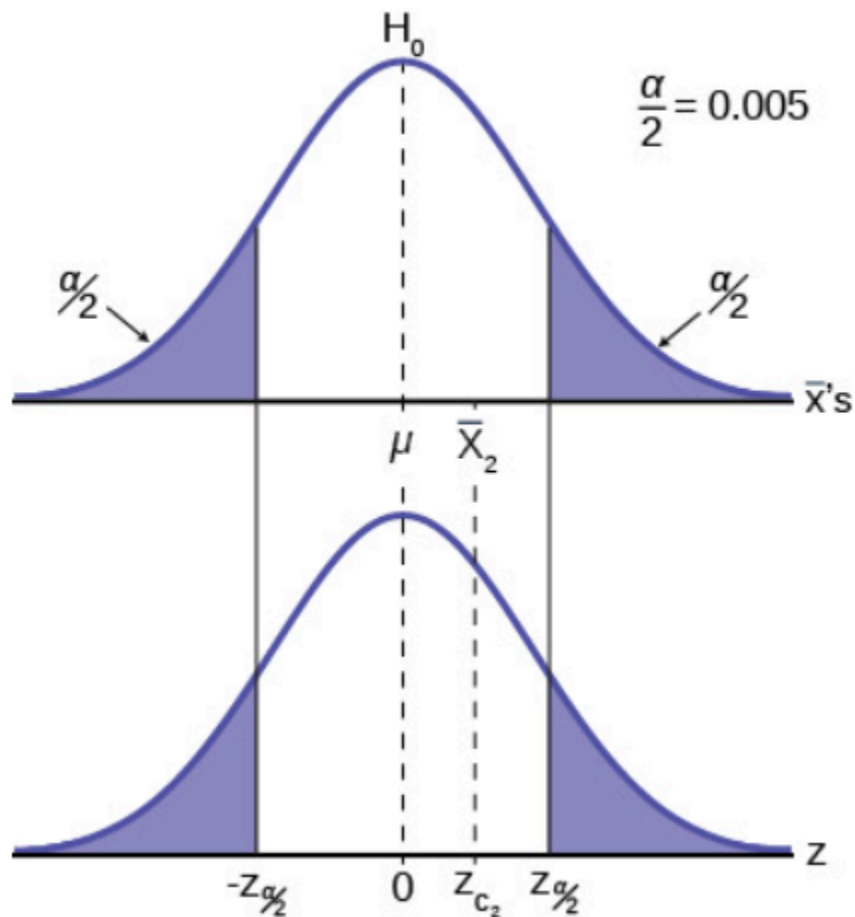


Figure 11

This gives us the decision rule for testing a hypothesis for a two-tailed test:

Table 7

Decision rule: two-tail testIf $|Z_c| < Z_{\frac{\alpha}{2}}$ then DO NOT REJECT H_0 If $|Z_c| > Z_{\frac{\alpha}{2}}$ then REJECT H_0

This rule will always be the same no matter what hypothesis we are testing or what formulas we are using to make the test. The only change will be change the Z_c to the appropriate symbol for the test statistic for the parameter being tested. Stating the decision rule another way: if the sample mean is unlikely to have come from the distribution with the hypothesized mean we cannot accept the null hypothesis. Here we define “unlikely” as having a probability less than alpha of occurring.

Hypothesis testing: Finding Critical Values



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Normal Distribution: Finding Critical Values of Z



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P-Value Approach

An alternative decision rule can be developed by calculating the probability that a sample mean could be found that would give a test statistic larger than the test statistic found from the current sample data assuming that the null hypothesis is true. Here the notion of “likely” and “unlikely” is defined by the probability of drawing a sample with a mean from a population with the hypothesized mean that is either larger or smaller than that found in the sample data. Simply stated, the p-value approach compares the desired significance level, α , to the p-value which is the probability of drawing a sample mean further from the hypothesized value than the actual sample mean. A large p-value calculated from the data indicates that we should not reject the null hypothesis. The smaller the p-value, the more unlikely the outcome, and the stronger the evidence is against the null hypothesis. We would reject the null hypothesis if the evidence is strongly against it. The relationship between the decision rule of comparing the calculated test statistics, Z_c , and the Critical Value, Z_{α} , and using the p-value can be seen in Figure 12.

problem. In any case, the value of α is the decision of the analyst. When you make a decision to reject or not reject H_0 , do as follows:

- If $\alpha > \text{p-value}$, cannot accept H_0 . The results of the sample data are significant. There is sufficient evidence to conclude that H_0 is an incorrect belief and that the **alternative hypothesis**, H_a , may be correct.
- If $\alpha \leq \text{p-value}$, cannot reject H_0 . The results of the sample data are not significant. There is not sufficient evidence to conclude that the alternative hypothesis, H_a , may be correct. In this case the status quo stands.
- When you “cannot reject H_0 ,” it does not mean that you should believe that H_0 is true. It simply means that the sample data have **failed** to provide sufficient evidence to cast serious doubt about the truthfulness of H_0 . Remember that the null is the status quo, and it takes high probability to overthrow the status quo. This bias in favor of the null hypothesis is what gives rise to the statement “tyranny of the status quo” when discussing hypothesis testing and the scientific method.

Both decision rules will result in the same decision, and it is a matter of preference which one is used.

What is a “P-Value?”



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One and Two-Tailed Tests

The discussion with the figures above was based on the null and alternative hypothesis presented in the first figure. This was called a two-tailed test because the alternative hypothesis allowed that the mean could have come from a population which was either larger or smaller than the hypothesized mean in the null hypothesis. This could be seen by the statement of the alternative hypothesis as $\mu \neq 100$, in this example.

It may be that the analyst has no concern about the value being “too” high or “too” low from the hypothesized value. If this is the case, it becomes a one-tailed test, and all of the alpha probability is placed in just one tail and not split into $\alpha/2$ as in the above case of a two-tailed test. Any test of a claim will be a one-tailed test. For example, a car manufacturer claims that their Model 17B provides gas mileage of greater than 25 miles per gallon. The null and alternative hypothesis would be:

$$H_0 : \mu \leq 25$$

$$H_a : \mu > 25$$

The claim would be in the alternative hypothesis. The burden of proof in hypothesis testing is carried in the alternative. This is because failing to reject the null, the status quo, must be accomplished with 90 or 95 percent confidence that it cannot be maintained. Said another way, we want to have only a 5 or 10 percent probability of making a Type I error, rejecting a good null; overthrowing the status quo.

This is a one-tailed test, and all of the alpha probability is placed in just one tail and not split into $\alpha/2$ as in the above case of a two-tailed test.

Figure 13 shows two possible cases and the form of the null and alternative hypothesis that give rise to them.

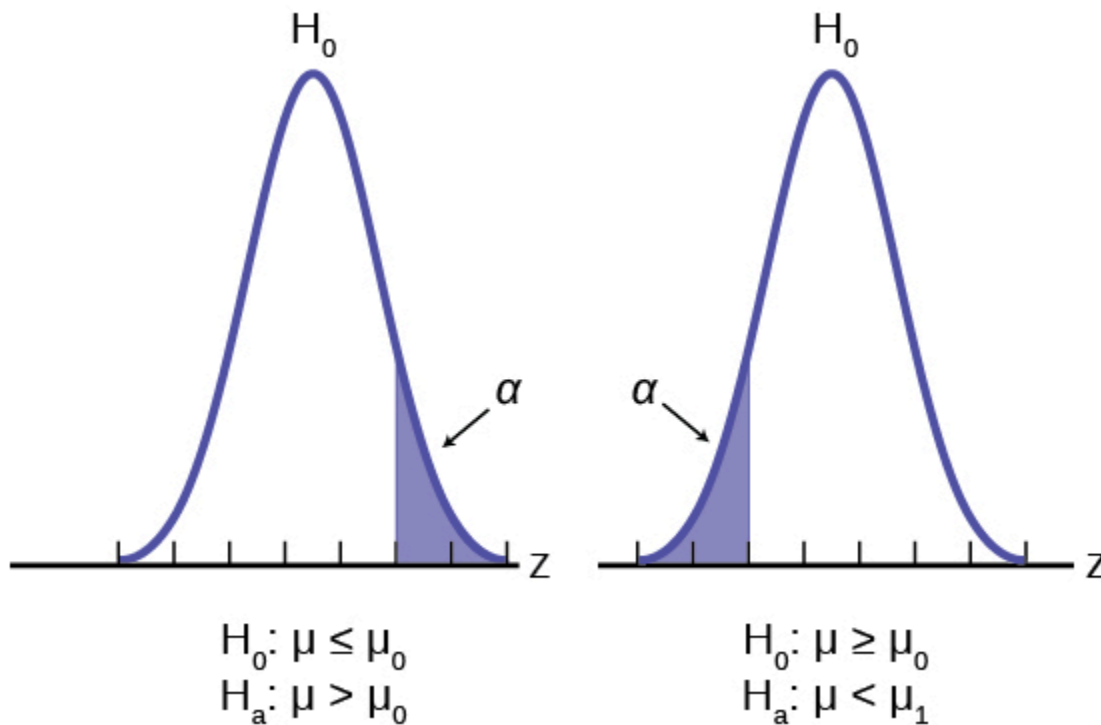


Figure 13

Where μ_0 is the hypothesized value of the population mean.

**Table 8: Test Statistics for Test of Means,
Varying Sample Size, Population Standard
Deviation Known or Unknown**

Sample size	Test statistic
$< 30 (\sigma \text{ unknown})$	$t_c = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$
$< 30 (\sigma \text{ known})$	$Z_c = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$
$> 30 (\sigma \text{ unknown})$	$Z_c = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$
$> 30 (\sigma \text{ known})$	$Z_c = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$

Effects of Sample Size on Test Statistic

In developing the confidence intervals for the mean from a sample, we found that most often we would not have the population standard deviation, σ . If the sample size were less than 30, we could simply substitute the point estimate for σ , the sample standard deviation, s , and use the student's t-distribution to correct for this lack of information.

When testing hypotheses, we are faced with this same problem and the solution is exactly the same. Namely: If the population standard deviation is unknown, and the sample size is less than 30, substitute s , the point estimate for the population standard deviation, σ , in the formula for the test statistic and use the student's t-distribution. All the formulas and figures above are unchanged except for this substitution and changing the Z distribution to the student's t-distribution on the graph. Remember that the student's t-distribution can only be computed knowing the proper degrees of freedom for the problem. In this case, the degrees of freedom is computed as before with confidence intervals: $df = (n - 1)$. The calculated t-value is compared to the t-value associated with the pre-set level of confidence required in the test, $t_{\alpha, df}$ found in the student's t tables. If we do not know σ , but the sample size is 30 or more, we simply substitute s for σ and use the normal distribution.

Table 8 summarizes these rules.

A Systematic Approach for Testing a Hypothesis

A systematic approach to hypothesis testing follows the following steps and in this order. This template will work for all hypotheses that you will ever test.

- Set up the null and alternative hypothesis. This is typically the hardest part of the process. Here the question being asked is reviewed. What parameter is being tested, a mean, a proportion, differences in means, etc. Is this a one-tailed test or two-tailed test?

- Decide the level of significance required for this particular case and determine the critical value. These can be found in the appropriate statistical table. The levels of confidence typical for businesses are 80, 90, 95, 98, and 99. However, the level of significance is a policy decision and should be based upon the risk of making a Type I error, rejecting a good null. Consider the consequences of making a Type I error.
Next, on the basis of the hypotheses and sample size, select the appropriate test statistic and find the relevant critical value: Z_{α} , t_{α} , etc. Drawing the relevant probability distribution and marking the critical value is always big help. Be sure to match the graph with the hypothesis, especially if it is a one-tailed test.
- Take a sample(s) and calculate the relevant parameters: sample mean, standard deviation, or proportion. Using the formula for the test statistic from above in step 2, now calculate the test statistic for this particular case using the parameters you have just calculated.
- Compare the calculated test statistic and the critical value. Marking these on the graph will give a good visual picture of the situation. There are now only two situations:
 - The test statistic is in the tail: Cannot Accept the null, the probability that this sample mean (proportion) came from the hypothesized distribution is too small to believe that it is the real home of these sample data.
 - The test statistic is not in the tail: Cannot Reject the null, the sample data are compatible with the hypothesized population parameter.
- Reach a conclusion. It is best to articulate the conclusion two different ways. First a formal statistical conclusion such as “With a 5 % level of significance we cannot accept the null hypotheses that the population mean is equal to XX (units of measurement)”. The second statement of the conclusion is less formal and states the action, or lack of action, required. If the formal conclusion was that above, then the informal one might be, “The machine is broken, and we need to shut it down and call for repairs.”

All hypotheses tested will go through this same process. The only changes are the relevant formulas and those are determined by the hypothesis required to answer the original question.

Hypothesis Testing: One Sample Z Test of the Mean (Critical Value Approach)



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Hypothesis Testing: t Test for the Mean (Critical Value Approach)



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Hypothesis Testing: 1 Sample Z Test of the Mean (Confidence Interval Approach)



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Hypothesis Testing: 1 Sample Z Test for Mean (P-Value Approach)



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Hypothesis Test for Proportions

Just as there were confidence intervals for proportions, or more formally, the population parameter p of the binomial distribution, there is the ability to test hypotheses concerning p .

The population parameter for the binomial is p . The estimated value (point estimate) for p is p' where $p' = x/n$, x is the number of successes in the sample and n is the sample size.

When you perform a hypothesis test of a population proportion p , you take a simple random sample from the population. The conditions for a **binomial distribution** must be met, which are: there are a certain number n of independent trials meaning random sampling, the outcomes of any trial are binary, success or failure, and each trial has the same probability of a success p . The shape of the binomial distribution needs to be similar to the shape of the normal distribution. To ensure this, the quantities np' and nq' must both be greater than five ($np' > 5$ and $nq' > 5$). In this case the binomial distribution of a sample (estimated) proportion can be approximated by the normal distribution with $\mu = np$ and $\sigma = \sqrt{npq}$. Remember that $q = 1 - p$. There is no distribution that can correct for this small sample bias and thus if these conditions are not met, we simply

cannot test the hypothesis with the data available at that time. We met this condition when we first were estimating confidence intervals for p .

Again, we begin with the standardizing formula modified because this is the distribution of a binomial.

$$Z = \frac{p' - p}{\sqrt{\frac{pq}{n}}}$$

Substituting p_0 , the hypothesized value of p , we have:

$$Z_c = \frac{p' - p_0}{\sqrt{\frac{p_0 q_0}{n}}}$$

This is the test statistic for testing hypothesized values of p , where the null and alternative hypotheses take one of the following forms:

Table 9

Two-tailed test	One-tailed test	One-tailed test
$H_0 : p = p_0$	$H_0 : p \leq p_0$	$H_0 : p \geq p_0$
$H_a : p \neq p_0$	$H_a : p > p_0$	$H_a : p < p_0$

The decision rule stated above applies here also: if the calculated value of Z_c shows that the sample proportion is “too many” standard deviations from the hypothesized proportion, the null hypothesis cannot be accepted. The decision as to what is “too many” is pre-determined by the analyst depending on the level of significance required in the test.

Hypothesis Testing: 1 Proportion using the Critical Value Approach



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HYPOTHESIS TESTING WITH TWO SAMPLES

Studies often compare two groups. For example, researchers are interested in the effect aspirin has in preventing heart attacks. Over the last few years, newspapers and magazines have reported various aspirin studies involving two groups. Typically, one group is given aspirin and the other group is given a placebo. Then, the heart attack rate is studied over several years.

There are other situations that deal with the comparison of two groups. For example, studies compare various diet and exercise programs. Politicians compare the proportion of individuals from different income brackets who might vote for them. Students are interested in whether

SAT or GRE preparatory courses really help raise their scores. Many business applications require comparing two groups. It may be the investment returns of two different investment strategies, or the differences in production efficiency of different management styles.

To compare two means or two proportions, you work with two groups. The groups are classified either as **independent** or **matched pairs**. **Independent groups** consist of two samples that are independent, that is, sample values selected from one population are not related in any way to sample values selected from the other population. **Matched pairs** consist of two samples that are dependent. The parameter tested using matched pairs is the population mean. The parameters tested using independent groups are either population means or population proportions of each group.

Comparing Two Independent Population Means

The comparison of two independent population means is very common and provides a way to test the hypothesis that the two groups differ from each other. Is the night shift less productive than the day shift, are the rates of return from fixed asset investments different from those from common stock investments, and so on? An observed difference between two sample means depends on both the means and the sample standard deviations. Very different means can occur by chance if there is great variation among the individual samples. The test statistic will have to account for this fact. The test comparing two independent population means with unknown and possibly unequal population standard deviations is called the Aspin-Welch t-test. The degrees of freedom formula we will see later was developed by Aspin-Welch.

When we developed the hypothesis test for the mean and proportions, we began with the Central Limit Theorem. We recognized that a sample mean came from a distribution of sample means, and sample proportions came from the sampling distribution of sample proportions. This made our sample parameters, the sample means and sample proportions, into random variables. It was important for us to know the distribution that these random variables came from. The Central Limit Theorem gave us the answer: the normal distribution. Our Z and t statistics came from this theorem. This provided us with the solution to our question of how to measure the probability that a sample mean came from a distribution with a particular hypothesized value of the mean or proportion. In both cases that was the question: what is the probability that the mean (or proportion) from our sample data came from a population distribution with the hypothesized value we are interested in?

Now we are interested in whether or not two samples have the same mean. Our question has not changed: Do these two samples come from the same population distribution? We recognize that we have two sample means, one from each set of data, and thus we have two random variables coming from two unknown distributions. To solve the problem, we create a new random variable, the difference between the sample means. This new random variable also has a distribution and, again, the Central Limit Theorem tells us that this new distribution is normally distributed, regardless of the underlying distributions of the original data. A graph may help to understand this concept.

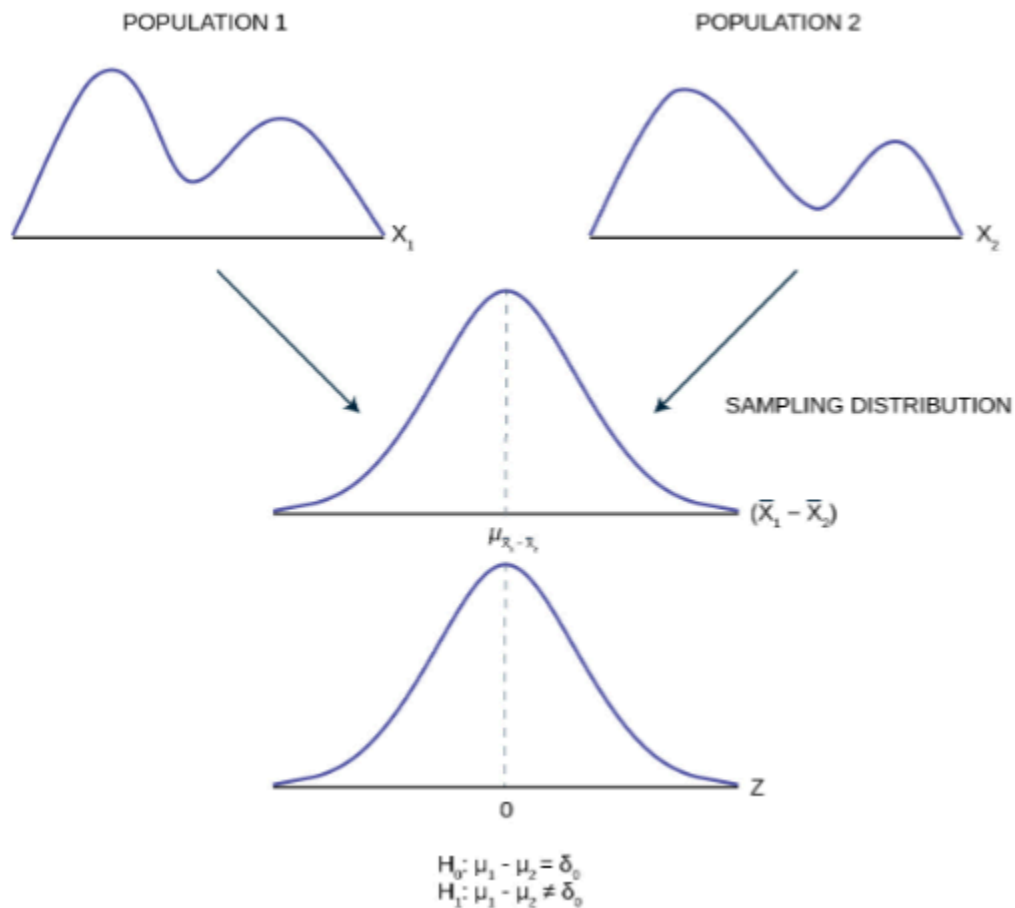


Figure 14

Pictured in Figure 14 are two distributions of data, X_1 and X_2 , with unknown means and standard deviations. The second panel shows the sampling distribution of the newly created random variable $(\bar{X}_1 - \bar{X}_2)$. This distribution is the theoretical distribution of many samples means from population 1 minus sample means from population 2. The Central Limit Theorem tells us that this theoretical sampling distribution of differences in sample means is normally distributed, regardless of the distribution of the actual population data shown in the top panel. Because the sampling distribution is normally distributed, we can develop a standardizing formula and calculate probabilities from the standard normal distribution in the bottom panel, the Z distribution.

The Central Limit Theorem, as before, provides us with the standard deviation of the sampling distribution, and further, that the expected value of the mean of the distribution of differences in sample means is equal to the differences in the population means. Mathematically this can be stated:

$$E(\mu_{x_1} - \mu_{x_2}) = \mu_1 - \mu_2$$

Because we do not know the population standard deviations, we estimate them using the two sample standard deviations from our independent samples. For the hypothesis test,

we calculate the estimated standard deviation, or **standard error**, of the **difference in sample means**, $\bar{X}_1 - \bar{X}_2$.

The standard error is:

$$\sqrt{\frac{(s_1)^2}{n_1} + \frac{(s_2)^2}{n_2}}$$

We remember that substituting the sample variance for the population variance when we did not have the population variance was the technique we used when building the confidence interval and the test statistic for the test of hypothesis for a single mean back in Confidence Intervals and Hypothesis Testing with One Sample. **The test statistic (t-score) is calculated as follows:**

$$t_c = \frac{(\bar{x} - \bar{x}_2) - \delta_0}{\sqrt{\frac{(s_1)^2}{n_1} + \frac{(s_2)^2}{n_2}}}$$

Where:

- s_1 and s_2 , the sample standard deviations, are estimates of σ_1 and σ_2 , respectively
- σ_1 and σ_2 are the unknown population standard deviations
- \bar{x}_1 and \bar{x}_2 are the sample means. μ_1 and μ_2 are the unknown population means.

The number of **degrees of freedom (df)** requires a somewhat complicated calculation. The df are not always a whole number. The test statistic above is approximated by the Student's t-distribution with df as follows:

$$df = \frac{\left(\frac{(s_1)^2}{n_1} + \frac{(s_2)^2}{n_2}\right)^2}{\left(\frac{1}{n_1 - 1}\right)\left(\frac{(s_1)^2}{n_1}\right)^2 + \left(\frac{1}{n_2 - 1}\right)\left(\frac{(s_2)^2}{n_2}\right)^2}$$

When both sample sizes n_1 and n_2 are 30 or larger, the Student's t approximation is very good. If each sample has more than 30 observations, then the degrees of freedom can be calculation as $n_1 + n_2 - 2$.

The format of the sampling distribution, differences in sample means, specifies that the format of the null and alternative hypothesis is:

$$H_0 : \mu_1 - \mu_2 = \delta_0$$

$$H_a : \mu_1 - \mu_2 \neq \delta_0$$

Where δ_0 is the hypothesized difference between the two means. If the question is simply

“is there any difference between the means?” then $\delta_0 = 0$ and the null and alternative hypotheses becomes:

$$H_0 : \mu_1 = \mu_2$$

$$H_a : \mu_1 \neq \mu_2$$

An example of when δ_0 might not be zero is when the comparison of the two groups requires a specific difference for the decision to be meaningful. Imagine that you are making a capital investment. You are considering changing from your current model machine to another. You measure the productivity of your machines by the speed they produce the product. It may be that a contender to replace the old model is faster in terms of product throughput but is also more expensive. The second machine may also have more maintenance costs, setup costs, etc. The null hypothesis would be set up so that the new machine would have to be better than the old one by enough to cover these extra costs in terms of speed and cost of production. This form of the null and alternative hypothesis shows how valuable this particular hypothesis test can be. For most of our work we will be testing simple hypotheses asking if there is any difference between the two-distribution means.

Hypothesis Testing – Two Population Means



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Two Population Means, One Tail Test



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Two Population Means, Two Tail Test



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Check Your Understanding: Hypothesis Testing (Two Population Means)



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Cohen's Standards for Small, Medium, and Large Effect Sizes

Cohen's d is a measure of "effect size" based on the differences between two means. Cohen's d, named for United States statistician Jacob Cohen, measures the relative strength of the differences between the means of two populations based on sample data. The calculated value of effect size is then compared to Cohen's standards of small, medium, and large effect sizes.

Table 10

Size of effect	d
Small	0.2
Medium	0.5
Large	0.8

Cohen's d is the measure of the difference between two means divided by the pooled standard deviation:

$$d = \frac{\bar{x}_1 - \bar{x}_2}{s_{\text{pooled}}} \text{ where } s_{\text{pooled}} = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

It is important to note that Cohen's d does not provide a level of confidence as to the magnitude of the size of the effect comparable to the other tests of hypothesis we have studied. The sizes of the effects are simply indicative.

Effect Size for a Significant Difference of Two Sample Means



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Test for Differences in Means: Assuming Equal Population Variances

Typically, we can never expect to know any of the population parameters, mean,

proportion, or standard deviation. When testing hypotheses concerning differences in means we are faced with the difficulty of two unknown variances that play a critical role in the test statistic. We have been substituting the sample variances just as we did when testing hypotheses for a single mean. And as we did before, we used a Student's t to compensate for this lack of information on the population variance. There may be situations, however, when we do not know the population variances, but we can assume that the two populations have the same variance. If this is true, then the pooled sample variance will be smaller than the individual sample variances. This will give more precise estimates and reduce the probability of discarding a good null. The null and alternative hypotheses remain the same, but the test statistic changes to:

$$t_c = \frac{(\bar{x}_1 - \bar{x}_2) - \delta_0}{\sqrt{S_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}}$$

Where S_p^2 is the pooled variance given by the formula:

$$S_p^2 = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

Example: A drug trial is attempted using a real drug and a pill made of just sugar. 18 people are given the real drug in hopes of increasing the production of endorphins. The increase in endorphins is found to be on average 8 micrograms per person, and the sample standard deviation is 5.4 micrograms. 11 people are given the sugar pill, and their average endorphin increase is 4 micrograms with a standard deviation of 2.4. From previous research on endorphins, it is determined that it can be assumed that the variances within the two samples can be assumed to be the same. Test at 5% to see if the population mean for the real drug had a significantly greater impact on the endorphins than the population mean with the sugar pill.

Solution:

First, we begin by designating one of the two groups Group 1 and the other Group 2. This will be needed to keep track of the null and alternative hypotheses. Let us set Group 1 as those who received the actual new medicine being tested and therefore Group 2 is those who received the sugar pill. We can now set up the null and alternative hypothesis as:

$$H_0 : \mu_1 \leq \mu_2$$

$$H_1 : \mu_1 > \mu_2$$

This is set up as a one-tailed test with the claim in the alternative hypothesis that the medicine will produce more endorphins than the sugar pill. We now calculate the test statistic which requires us to calculate the pooled variance, S_p^2 using the formula above.

$$t_c = \frac{(\bar{x}_1 - \bar{x}_2) - \delta_0}{\sqrt{S_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}} = \frac{(8 - 4) - 0}{\sqrt{20.4933 \left(\frac{1}{18} + \frac{1}{11} \right)}} = 2.31$$

t_{α} , allows us to compare the test statistic and the critical value.

$$t_{\alpha} = 1.703 \text{ at } df = n_1 + n_2 - 2 = 18 + 11 - 2 = 27$$

The test statistic is clearly in the tail, 2.31 is larger than the critical value of 1.703, and therefore we cannot maintain the null hypothesis. Thus, we conclude that there is significant evidence at the 95% level of confidence that the new medicine produces the effect desired.

Two Population Means with Known Standard Deviations

Even though this situation is not likely (knowing the population standard deviations is very unlikely), the following example illustrates hypothesis testing for independent means with known population standard deviations. The sampling distribution for the difference between the means is normal in accordance with the central limit theorem. The random variable is $\bar{X}_1 - \bar{X}_2$. The normal distribution has the following format:

The standard deviation is:

$$\sqrt{\frac{(\sigma_1)^2}{n_1} + \frac{(\sigma_2)^2}{n_2}}$$

The test statistic (z-score) is:

$$Z_c = \frac{(\bar{x}_1 - \bar{x}_2) - \delta_0}{\sqrt{\frac{(\sigma_1)^2}{n_1} + \frac{(\sigma_2)^2}{n_2}}}$$

Two Population Means, One Tail Test



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Two Population Means, Two Tail Test



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Check Your Understanding: Two Population Means



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Matched or Paired Samples

In most cases of economic or business data we have little or no control over the process of how the data are gathered. In this sense the data are not the result of a planned controlled experiment. In some cases, however, we can develop data that are part of a controlled experiment. This situation occurs frequently in quality control situations. Imagine that the production rates of two machines built to the same design, but at different manufacturing plants, are being tested for differences in some production metric such as speed of output or meeting some production specification such as strength of the product. The test is the same in format to what we have been testing, but here we can have matched pairs for which we can test if differences exist. Each observation has its matched pair against which differences are calculated. First, the differences in the metric to be tested between the two lists of observations must be calculated, and this is typically labeled with the letter “d.” Then, the average of these matched differences, \bar{X}_d is calculated as is its standard deviation, S_d . We expect that the standard deviation of the differences of the matched pairs will be smaller than unmatched pairs because presumably fewer differences should exist because of the correlation between the two groups.

When using a hypothesis test for matched or paired samples, the following characteristics may be present:

1. Simple random sampling is used.
2. Sample sizes are often small.
3. Two measurements (samples) are drawn from the same pair of individuals or objects.
4. Differences are calculated from the matched or paired samples.
5. The differences form the sample that is used for the hypothesis test.
6. Either the matched pairs have differences that come from a population that is normal or the number of differences is sufficiently large so that distribution of the sample mean of differences is approximately normal.

In a hypothesis test for matched or paired samples, subjects are matched in pairs and differences are calculated. The differences are the data. The population mean for the differences, μ_d , is then tested using a Student’s t-test for a single population mean with

$n - 1$ degrees of freedom, where n is the number of differences, that is, the number of pairs not the number of observations.

The null and alternative hypotheses for this test are:

$$H_0 : \mu_d = 0$$

$$H_a : \mu_d \neq 0$$

The test statistic is:

$$t_c = \frac{\bar{x}_d - \mu_d}{\left(\frac{s_d}{\sqrt{n}}\right)}$$

Two Population Means, One Tail Test, Matched Sample



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Two Population Means, One Tail test, Matched Sample (Hypothesized Test Different from Zero)



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Matched Sample, Two Tail Test



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Check Your Understanding: Matched Sample, Two Tail Test



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Comparing Two Independent Population Proportions

When conducting a hypothesis test that compares two independent population proportions, the following characteristics should be present:

1. The two independent samples are random samples are independent.
2. The number of successes is at least five, and the number of failures is at least five, for each of the samples.
3. Growing literature states that the population must be at least ten or even perhaps 20 times the size of the sample. This keeps each population from being over-sampled and causing biased results.

Comparing two proportions, like comparing two means, is common. If two estimated proportions are different, it may be due to a difference in the populations or it may be due to chance in the sampling. A hypothesis test can help determine if a difference in the estimated proportions reflects a difference in the two population proportions.

Like the case of differences in sample means, we construct a sampling distribution for differences in sample proportions: $(p'_A - p'_B)$ where $p'_A = \frac{X_A}{n_A}$ and $p'_B = \frac{X_B}{n_B}$ are the sample proportions for the two sets of data in question. X_A and X_B are the number of successes in each sample group respectively, and n_A and n_B are the respective sample sizes from the two groups. Again, we go the Central Limit Theorem to find the distribution of this sampling distribution for the differences in sample proportions. And again, we find that this sampling distribution, like the one's past, are normally distributed as proved by the Central Limit Theorem, as see in Figure 15.

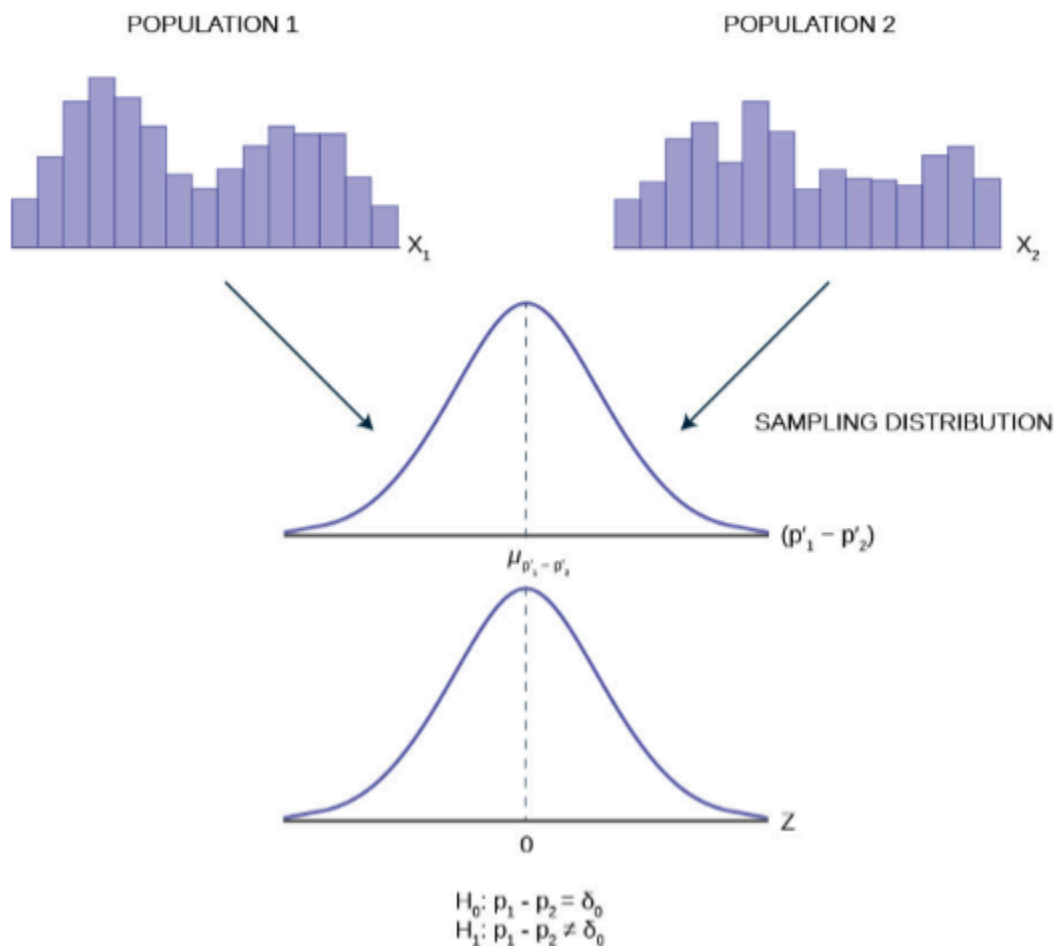


Figure 15

Generally, the null hypothesis allows for the test of a difference of a particular value, δ_0 , just as we did for the case of difference in means.

$$H_0 : p_1 - p_2 = \delta_0$$

$$H_1 : p_1 - p_2 \neq \delta_0$$

Most common, however, is the test that the two proportions are the same. That is,

$$H_0 : p_A = p_B$$

$$H_a : p_A \neq p_B$$

To conduct the test, we use a pooled proportion, p_c .

The pooled proportion is calculated as follows:

$$p_c = \frac{x_A + x_B}{n_A + n_B}$$

The test statistic (z-score) is:

$$Z_c = \frac{(p'_A - p'_B) - \delta_0}{\sqrt{p_c(1-p_c)\left(\frac{1}{n_A} + \frac{1}{n_B}\right)}}$$

Where δ_0 is the hypothesized differences between the two proportions and p_c is the pooled variance from the formula above.

Two Population Proportions, One Tail Test



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Two Population Proportion, Two Tail Test



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Check Your Understanding: Comparing Two Independent Population Proportions



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THE CHI-SQUARE DISTRIBUTION

Have you ever wondered if lottery winning numbers were evenly distributed or if some numbers occurred with a greater frequency? How about if the types of movies people preferred were different across different age groups? What about if a coffee machine was dispensing approximately the same amount of coffee each time? You could answer these questions by conducting a hypothesis test.

You will now study a new distribution, one that is used to determine the answers to such questions. This distribution is called the chi-square distribution.

In this section, you will learn the three major applications of the chi-square distribution:

1. The test of a single variance, which tests variability, such as in the coffee example
2. The goodness-of-fit test, which determines if data fit a particular distribution, such as in the lottery example
3. the test of independence, which determines if events are independent, such as in the movie example

Facts About the Chi-Square Distribution

The notation for the **chi-square distribution** is:

$$\chi \sim X_{df}^2$$

Where df = degrees of freedom which depends on how chi-square is being used. (If you want to practice calculating chi-square probabilities then use $df = n - 1$. The degrees of freedom for the three major uses are each calculated differently.)

For the χ^2 distribution, the population mean is $\mu = df$ and the population standard deviation is $\sigma = \sqrt{d(df)}$.

The random variable is shown as χ^2 .

The random variable for a chi-square distribution with k degrees of freedom is the sum of k independent, squared standard normal variables.

$$\chi^2 = (Z_1)^2 + (Z_2)^2 + \dots + (Z_k)^2$$

1. The curve is nonsymmetrical and skewed to the right.
2. There is a different chi-square curve for each df .

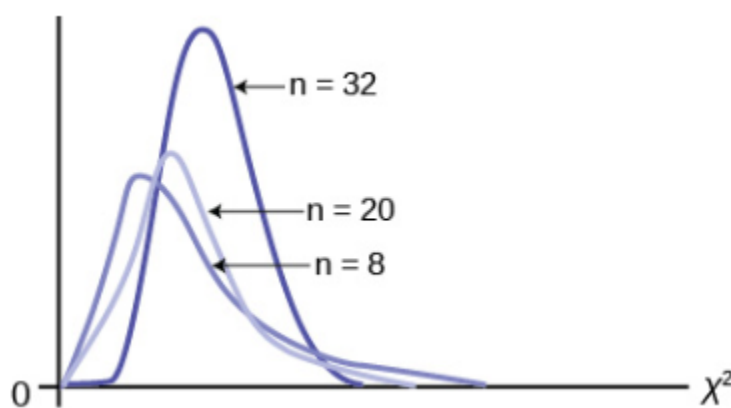


Figure 16

3. The test statistic for any test is always greater than or equal to zero.
4. When $df > 90$, the chi-square curve approximates the normal distribution. For

$\chi \sim \chi_{1,000}^2$ the mean, $\mu = df = 1,000$ and the standard deviation, $\sigma = \sqrt{2(1,000)} = 44.7$. Therefore, $\chi \sim N(1,000, 44.7)$, approximately.

5. The mean, μ , is located just to the right of the peak.

Test of a Single Variance

Thus far our interest has been exclusively on the population parameter μ or it is counterpart in the binomial, p . Surely the mean of a population is the most critical piece of information to have, but in some cases, we are interested in the variability of the outcomes of some distribution. In almost all production processes quality is measured not only by how closely the machine matches the target, but also the variability of the process. If one were filling bags with potato chips not only would there be interest in the average weight of the bag, but also how much variation there was in the weights. No one wants to be assured that the average weight is accurate when their bag has no chips. Electricity voltage may meet some average level, but great variability, spikes, can cause serious damage to electrical machines, especially computers. I would not only like to have a high mean grade in my classes, but also low variation about this mean. In short, statistical tests concerning the variance of a distribution have great value and many applications.

A **test of a single variance** assumes that the underlying distribution is **normal**. The null and alternative hypotheses are stated in terms of the **population variance**. The test statistic is:

$$\chi_c^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

Where:

- n = the total number of observations in the sample data
- s^2 = sample variance
- σ_0^2 = hypothesized value of the population variance
- $H_0 : \sigma^2 = \sigma_0^2$
- $H_a : \sigma^2 \neq \sigma_0^2$

You may think of s as the random variable in this test. The number of degrees of freedom is $df = n - 1$. A test of a single variance may be right-tailed, left-tailed, or two-tailed. The example below will show you how to set up the null and alternative hypotheses. The null and alternative hypotheses contain statements about the population variance.

Example: Math instructors are not only interested in how their students do on exams,

on average, but how the exam scores vary. To many instructors, the variance (or standard deviation) may be more important than the average.

Suppose a math instructor believes that the standard deviation for his final exam is five points. One of his best students thinks otherwise. The student claims that the standard deviation is more than five points. If the student were to conduct a hypothesis test, what would the null and alternative hypotheses be?

Solution: Even though we are given the population standard deviation, we can set up the test using the population variance as follows:

$$H_0 : \sigma^2 \leq 5^2$$

$$H_a : \sigma^2 > 5^2$$

Single Population Variances, One-Tail Test



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Check Your Understanding: Test of a Single Variance



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Goodness-Of-Fit Test

In this type of hypothesis test, you determine whether the data “**fit**” a particular distribution or not. For example, you may suspect your unknown data fit a binomial distribution. You use a chi-square test (meaning the distribution for the hypothesis test is chi-square) to determine if there is a fit or not. **The null and the alternative hypotheses for this test may be written in sentences or may be stated as equations or inequalities.**

The test statistic for a goodness-of-fit test is:

$$\sum_k \frac{(O-E)^2}{E}$$

Where:

- **O = observed values** (data)
- **E = expected values** (from theory)
- **k = the number of different data cells or categories**

The observed values are the data values, and the expected values are the values you would expect to get if the null hypothesis were true. There are n terms of the form

$$\frac{(O-E)^2}{E}.$$

The number of degrees of freedom is $df = (\text{number of categories} - 1)$.

The goodness-of-fit test is almost always right-tailed. If the observed values and the corresponding expected values are not close to each other, then the test statistic can get very large and will be way out in the right tail of the chi-square curve.

Note: The number of expected values inside each cell needs to be at least five in order to use this test.

Chi-Square Statistic for Hypothesis Testing



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Chi-Square Goodness-of-Fit Example



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Check Your Understanding: Goodness-of-Fit Test



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Test of Independence

Tests of independence involve using a **contingency table** of observed (data) values.

The test statistic of a **test of independence** is similar to that of a goodness-of-fit test:

$$\sum_{i \cdot j} \frac{(O - E)^2}{E}$$

Where:

- O = observed values
- E = expected values
- i = the number of rows in the table
- j = the number of columns in the table

There are $i \cdot j$ terms of the form $\frac{(O - E)^2}{E}$

A test of independence determines whether two factors are independent or not.

Note: The expected value inside each cell needs to be at least five in order for you to use this test.

The test of independence is always right tailed because of the calculation of the test statistic. If the expected and observed values are not close together, then the test statistic is very large and way out in the right tail of the chi-square curve, as it is in a goodness-of-fit.

The number of degrees of freedom for the test of independence is:

$$df = (\text{Number of columns} - 1) (\text{number of rows} - 1)$$

The following formula calculates the expected number (E):

$$E = \frac{(\text{row total})(\text{column total})}{\text{total number surveyed}}$$

Simple Explanation of Chi-Squared



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Chi-Square Tet for Association (independence)



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Check Your Understanding: Test of Independence



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Test for Homogeneity

The goodness-of-fit test can be used to decide whether a population fits a given distribution, but it will not suffice to decide whether two populations follow the same unknown distribution. A different test, called the **test of homogeneity**, can be used to draw a conclusion about whether two populations have the same distribution. To calculate the test statistic for a test for homogeneity, follow the same procedure as with the test of independence.

Note: The expected value inside each cell needs to be at least five in order for you to use this test.

Hypotheses

H_0 :: The distribution of the two populations is the same.

H_a : The distributions of the two population are not the same.

Test Statistic

Use a χ^2 test statistic. It is computed in the same way as the test for independence.

Degrees of Freedom (df)

$$df = \text{number of columns} - 1$$

Requirements

All values in the table must be greater than or equal to five

Common uses

Comparing two populations. For example: men vs. women, before vs. after, east vs. west. The variable is categorical with more than two possible response values.

Introduction to the Chi-Square Test for Homogeneity



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Check Your Understanding: Test for Homogeneity



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Comparison of the Chi-Square Tests

Above the χ^2 test statistic was used in three different circumstances. The following bulleted list is a summary of which χ^2 test is the appropriate one to use in different circumstances.

Goodness-of-fit: Use the goodness-of-fit test to decide whether a population with an unknown distribution “fits” a known distribution. In this case there will be a single qualitative survey question or a single outcome of an experiment from a single

population. Goodness-of-Fit is typically used to see if the population is uniform (all outcomes occur with equal frequency), the population is normal, or the population is the same as another population with a known distribution. The null and alternative hypotheses are:

H_0 : The population fits given distribution

H_a : The population does not fit the given distribution

Independence: Use the test for independence to decide whether two variables (factors) are independent or dependent. In this case there will be two qualitative survey questions or experiments and a contingency table will be constructed. The goal is to see if the two variables are unrelated (independent) or related (dependent). The null and alternative hypotheses are:

H_0 : The two variables (factors) are independent

H_a : The two variables (factors) are dependent

Homogeneity: Use the test for homogeneity to decide if two populations with unknown distributions have the same distribution as each other. In this case there will be a single qualitative survey question or experiment given to two different populations. The null and alternative hypotheses are:

H_0 : The two populations follow the same distribution

H_a : The two populations have different distributions

F DISTRIBUTION AND ONE-WAY ANOVA

Many statistical applications in psychology, social science, business administration, and the natural sciences involve several groups. For example, an environmentalist is interested in knowing if the average amount of pollution varies in several bodies of water. A sociologist is interested in knowing if the amount of income a person earns varies according to his or her upbringing. A consumer looking for a new car might compare the average gas mileage of several models.

For hypothesis tests comparing averages among more than two groups, statisticians have developed a method called “Analysis of Variance” (abbreviated ANOVA). In this chapter, you will study the simplest form of ANOVA called single factor or one-way ANOVA. You will also study the F distribution, used for one-way ANOVA, and the test for differences between two variances. This is just a very brief overview of one-way ANOVA. One-Way ANOVA, as it is presented here, relies heavily on a calculator or computer.

Test of Two Variances

This chapter introduces a new probability density function, the F distribution. This distribution is used for many applications including ANOVA and for testing equality

across multiple means. We begin with the F distribution and the test of hypothesis of differences in variances. It is often desirable to compare two variances rather than two averages. For instance, college administrators would like two college professors grading exams to have the same variation in their grading. In order for a lid to fit a container, the variation in the lid and the container should be approximately the same. A supermarket might be interested in the variability of check-out times for two checkers. In finance, the variance is a measure of risk and thus an interesting question would be to test the hypothesis that two different investment portfolios have the same variance, the volatility.

In order to perform a F test of two variances, it is important that the following are true:

1. The populations from which the two samples are drawn are approximately normally distributed.
2. The two populations are independent of each other.

Unlike most other hypothesis tests in this Module, the F test for equality of two variances is very sensitive to deviations from normality. If the two distributions are not normal, or close, the test can give a biased result for the test statistic.

Suppose we sample randomly from two independent normal populations. Let σ_1^2 and σ_2^2 be the unknown population variances and s_1^2 and s_2^2 be the sample variances. Let the sample sizes be n_1 and n_2 . Since we are interested in comparing the two sample variances, we use the F ratio:

$$F = \frac{\left[\frac{s_1^2}{\sigma_1^2} \right]}{\left[\frac{s_2^2}{\sigma_2^2} \right]}$$

F has the distribution $F \sim F(n_1 - 1, n_2 - 1)$

Where $n_1 - 1$ are the degrees of freedom for the numerator and $n_2 - 1$ are the degrees of freedom for the denominator.

If the null hypothesis is $\sigma_1^2 = \sigma_2^2$, then the F Ratio, test statistic, becomes

$$F_c = \frac{\left[\frac{s_1^2}{\sigma_1^2} \right]}{\left[\frac{s_2^2}{\sigma_2^2} \right]} = \frac{s_1^2}{s_2^2}$$

The various forms of the hypotheses tested are:

Two-Tailed Test	One-Tailed Test	One-Tailed Test
$H_0 : \sigma_1^2 = \sigma_2^2$	$H_0 : \sigma_1^2 \leq \sigma_2^2$	$H_0 : \sigma_1^2 \geq \sigma_2^2$
$H_1 : \sigma_1^2 \neq \sigma_2^2$	$H_1 : \sigma_1^2 > \sigma_2^2$	$H_1 : \sigma_1^2 < \sigma_2^2$

A more general form of the null and alternative hypothesis for a two tailed test would be:

$$H_0 : \frac{\sigma_1^2}{\sigma_2^2} = \delta_0$$

$$H_a : \frac{\sigma_1^2}{\sigma_2^2} \neq \delta_0$$

Where if $\delta_0 = 1$ it is a simple test of the hypothesis that the two variances are equal. This form of the hypothesis does have the benefit of allowing for tests that are more than for simple differences and can accommodate tests for specific differences as we did for differences in means and differences in proportions. This form of the hypothesis also shows the relationship between the F distribution and the χ^2 : the F is a ratio of two chi squared distributions a distribution we saw in the last chapter. This is helpful in determining the degrees of freedom of the resultant F distribution.

If the two populations have equal variances, then s_1^2 and s_2^2 are close in value and the test statistic, $F_c = \frac{s_1^2}{s_2^2}$ is close to one. But if the two population variances are very different, s_1^2 and s_2^2 tend to be very different, too. Choosing s_1^2 as the larger sample variance causes the ratio $\frac{s_1^2}{s_2^2}$ to be greater than one. If s_1^2 and s_2^2 are far apart, then $F_c = \frac{s_1^2}{s_2^2}$ is a large number.

Therefore, if F is close to one, the evidence favors the null hypothesis (the two population variances are equal). But if F is much larger than one, then the evidence is against the null hypothesis. In essence, we are asking if the calculated F statistic, test statistic, is significantly different from one.

To determine the critical points, we have to find $F_{\alpha, df1, df2}$ (link for F table). This F table has values for various levels of significance from 0.1 to 0.001 designated as “p” in the first column. To find the critical value choose the desired significance level and follow down and across to find the critical value at the intersection of the two different degrees of freedom. The F distribution has two different degrees of freedom, one associated with the numerator, $df1$, and one associated with the denominator, $df2$ and to complicate matters, the F distribution is not symmetrical and changes the degree of skewness as the degrees of freedom of change. The degrees of freedom in the numerator is $n_1 - 1$, where n_1 is the sample size for group 1, and the degrees of freedom in the denominator is $n_2 - 1$, where n_2 is the sample size for group 2. $F_{\alpha, df1, df2}$ will give the critical value on the **upper** end of the F distribution.

To find the critical value for the lower end of the distribution, reverse the degrees of freedom and divide the F-value from the table into one.

- Upper tail critical value: $F_{\alpha, df1, df2}$
- Lower tail critical value: $1 / F_{\alpha, df2, df1}$

When the calculated value of F is between the critical values, not in the tail, we cannot reject the null hypothesis that the two variances came from a population with the same variance. If the calculated F-value is in either tail we cannot accept the null hypothesis just as we have been doing for all of the previous tests of hypothesis.

An alternative way of finding the critical values of the F distribution makes the use of the F-table easier. We note in the F-table that all the values of F are greater than one therefore the critical F value for the left-hand tail will always be less than one because to find the critical value on the left tail we divide an F value into the number one as shown above. We also note that if the sample variance in the numerator of the test statistic is larger than the sample variance in the denominator, the resulting F value will be greater than one. The shorthand method for this test is thus to be sure that the larger of the two sample variances is placed in the numerator to calculate the test statistic. This will mean that only the right-hand tail critical value will have to be found in the F-table.

Hypothesis Test Two Population Variances



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Check Your Understanding: F Distribution



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One-Way ANOVA

The purpose of a one-way ANOVA test is to determine the existence of a statistically

significant difference among several group means. The test actually uses **variances** to help determine if the means are equal or not. In order to perform a one-way ANOVA test, there are five basic **assumptions** to be fulfilled:

1. Each population from which a sample is taken is assumed to be normal.
2. All samples are randomly selected and independent.
3. The populations are assumed to have **equal standard deviations (or variances)**.
4. The factor is a categorical variable.
5. The response is a numerical variable.

The Null and Alternative Hypotheses

The null hypothesis is simply that all the group population means are the same. The alternative hypothesis is that at least one pair of means is different. For example, if there are k groups:

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \dots \mu_k$$

H_a : At least two of the group means $\mu_1, \mu_2, \mu_3, \dots, \mu_k$ are not equal. That is, $\mu_i \neq \mu_j$ for some $i \neq j$.

The graphs, a set of box plots representing the distribution of values with the group means indicated by a horizontal line through the box, help in the understanding of the hypothesis test. In the first graph (red box plots), $H_0 : \mu_1 = \mu_2 = \mu_3$ and the three populations have the same distribution if the null hypothesis is true. The variance of the combined data is approximately the same as the variance of the populations.

If the null hypothesis is false, then the variance of the combined data is larger which is caused by the different means as shown in the second graph (green box plots).

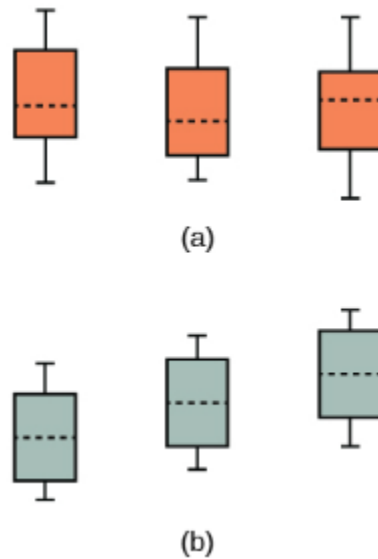


Figure 17: (a) H_0 is true. All means are the same; the differences are due to random variation. (b) H_0 is not true. All means are not the same; the differences are too large to be due to random variation.

ANOVA



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The F Distribution and the F-Ratio

The distribution used for the hypothesis test is a new one. It is called the F distribution, invented by George Snedecor but named in honor of Sir Ronald Fisher, an English statistician. The F statistic is a ratio (a fraction). There are two sets of degrees of freedom: one for the numerator and one for the denominator.

For example, if F follows an F distribution and the number of degrees of freedom for the numerator is four, and the number of degrees of freedom for the denominator is ten, then $F \sim F_{4,10}$.

To calculate the **F ratio**, two estimates of the variance are made.

1. **Variance between samples:** An estimate of σ^2 that is the variance of the sample means multiplied by n (when the sample sizes are the same). If the samples are different sizes, the variances between samples is weighted to account for the different sample sizes. The variances is also called **variance due to treatment or explained variation**.

2. **Variance within samples:** An estimate of σ^2 that is the average of the sample variances (also known as a pooled variance). When the sample sizes are different, the variance within samples is weighted. The variance is also called the **variation due to error or unexplained variation**.

- SS_{between} = the **sum of squares** that represents the variation among the different samples
- SS_{within} = the sum of squares that represents the variation within samples that is due to chance.

To find a “sum of squares” means to add together squared quantities that, in some cases, may be weighted.

MS means “**mean square**.” MS_{between} is the variance between groups, and MS_{within} is the variance within groups.

Calculation of Sum of Squares and Mean Square

- k = the number of different groups
- n_j = the size of the j_{th} group
- s_j = the sum of the values in the j_{th} group
- n = total number of all the values combined (total sample size: $\sum n_j$)
- x = one value: $\sum s = \sum s_j$
- Sum of squares of all values from every group combined: $\sum x^2$
- Between group variability: $SS_{\text{total}} = \sum x^2 - \frac{(\sum x)^2}{n}$
- Total sum of squares: $\sum x^2 - \frac{(\sum x)^2}{n}$
- Explains variation: sum of squares representing variation among the different samples:

$$SS_{\text{between}} = \sum \left[\frac{(s_j)^2}{n_j} - \frac{(\sum s_j)^2}{n} \right]$$
- Unexplained variation: sum of squares represent variation within samples due to change:

$$SS_{\text{within}} = SS_{\text{total}} - SS_{\text{between}}$$
- df 's for different groups (df 's for the numerator): $df = k - 1$
- Equation for errors within samples (df 's for the denominator): $df_{\text{within}} = n - k$

- Mean square (variance estimate) explained by the different groups:

$$MS_{\text{between}} = \frac{SS_{\text{between}}}{df_{\text{between}}}$$

- Mean square (variance estimate) that is due to chance (unexplained):
(unexplained): $MS_{\text{within}} = \frac{SS_{\text{within}}}{df_{\text{within}}}$

MS_{between} and MS_{within} can be written as follows:

- $MS_{\text{between}} = \frac{SS_{\text{between}}}{df_{\text{between}}} = \frac{SS_{\text{between}}}{k-1}$
- $MS_{\text{within}} = \frac{SS_{\text{within}}}{df_{\text{within}}} = \frac{SS_{\text{within}}}{n-k}$

The one-way ANOVA test depends on the fact that MS_{between} can be influenced by population differences among means of the several groups. Since MS_{within} compares values of each group to its own group mean, the fact that group means might be different does not affect MS_{within} .

The null hypothesis says that all groups are samples from populations having the same normal distribution. The alternate hypothesis says that at least two of the sample groups come from populations with different normal distributions. If the null hypothesis is true, MS_{between} and MS_{within} should both estimate the same value.

Note: The null hypothesis says that all the group population means are equal. The hypothesis of equal means implies that the populations have the same normal distribution, because it is assumed that the populations are normal and that they have equal variances.

F-Ration or F Statistic

$$F = \frac{MS_{\text{between}}}{MS_{\text{within}}}$$

If MS_{between} and MS_{within} estimate the same value (following the belief that H_0 is true), then the F-ratio should be approximately equal to one. Mostly, just sampling errors would contribute to variations away from one. As it turns out, MS_{between} consists of the population variance plus a variance produced from the differences between the samples. MS_{within} is an estimate of the population variance. Since variances are always positive, if

the null hypothesis is false, MS_{between} will generally be larger than MS_{within} . Then the F-ratio will be larger than one. However, if the population effect is small, it is not unlikely that MS_{within} will be larger in a given sample.

The foregoing calculations were done with groups of different sizes. If the groups are the same size, the calculations simplify somewhat, and the F-ratio can be written as:

F-Ratio Formula when the groups are the same size

$$F = \frac{n \cdot s_{\bar{x}}^2}{s_{\text{pooled}}^2}$$

Where:

- n = the sample size
- $df_{\text{numerator}} = k - 1$
- $df_{\text{denominator}} = n - k$
- s_{pooled}^2 = the mean of the sample variances (pooled variance)
- $s_{\bar{x}}^2$ = the variance of the sample means.

Data are typically put into a table for easy viewing. One-Way ANOVA results are often displayed in this manner by computer software.

Table 12

Source of variation	Sum of squares (SS)	Degrees of freedom (df)	Mean square (MS)	F
Factor (Between)	SS(Factor)	$k - 1$	$MS(\text{Factor}) = SS(\text{Factor}) / (k - 1)$	$F = MS(\text{Factor}) / MS(\text{Error})$
Error (Within)	SS(Error)	$n - k$	$MS(\text{Error}) = SS(\text{Error}) / (n - k)$	
Total	SS(Total)	$n - 1$		

Example: Three different diet plans are to be tested for mean weight loss. The entries in the table are the weight losses for the different plans. The one-way ANOVA results are shown in Table 13.

Table 13

Plan 1: $n_1 = 4$	Plan 2: $n_2 = 3$	Plan 3: $n_3 = 3$
5	3.5	8
4.5	7	4
4		3.5
3	4.5	

$$s_1 = 16.5, s_2 = 15, s_3 = 15.5$$

Following are the calculations needed to fill in the one-way ANOVA table. The table is used to conduct a hypothesis test.

$$SS_{\text{between}} = \sum \left[\frac{(s_j)^2}{n_j} \right] - \frac{(\sum s_j)^2}{n}$$

$$= \frac{s_1^2}{4} + \frac{s_2^2}{3} + \frac{s_3^2}{3} - \frac{(s_1 + s_2 + s_3)^2}{10}$$

Where $n_1 = 4, n_2 = 3, n_3 = 3$ and $n = n_1 + n_2 + n_3 = 10$

$$= \frac{(16.5)^2}{4} + \frac{(15)^2}{3} + \frac{(15.5)^2}{3} - \frac{(16.5 + 15 + 15.5)^2}{10}$$

$$SS_{\text{between}} = 2.2458$$

$$S_{\text{total}} = \sum x^2 - \frac{(\sum x)^2}{n}$$

$$= (5^2 + 4.5^2 + 4^2 + 3^2 + 3.5^2 + 7^2 + 4.5^2 + 8^2 + 4^2 + 3.5^2)$$

$$- \frac{5^2 + 4.5^2 + 4^2 + 3^2 + 3.5^2 + 7^2 + 4.5^2 + 8^2 + 4^2 + 3.5^2}{10}$$

$$= 244 - \frac{47^2}{10} = 244 - 220.9$$

$$SS_{\text{total}} = 23.1$$

$$SS_{\text{within}} = SS_{\text{total}} - SS_{\text{between}}$$

$$= 23.1 - 2.2458$$

$$SS_{\text{within}} = 20.8542$$

Table 14

Source of variation	Sum of squares (SS)	Degrees of freedom (df)	Mean square (MS)	F
Factor (Between)	SS(Factor)	$k - 1$	$MS(\text{Factor})$	$F =$
	= SS(Between)	= 3 groups - 1	= $SS(\text{Factor}) / (k - 1)$	$MS(\text{Factor}) / MS(\text{Error})$
	= 2.2458	= 2	= $2.2458 / 2$	= $1.1229 / 2.9792$
Error (Within)				= 0.3769
	SS(Error)	$n - k$	$MS(\text{Error})$	
	= SS(Within)	= 10 total data	= $SS(\text{Error}) / (n - k)$	
Total		- 3 groups	= $20.8542 / 7$	
		= 7	= 2.9792	
	SS(Total)	$n - 1$		
	= $2.2458 + 20.8542$	= 10 total data		
	= 23.1	- 1		
		= 9		

The one-way ANOVA hypothesis test is always right-tailed because larger F-values are way out in the right tail of the F-distribution and tend to make us reject H_0 .

Notation

The notation for the F distribution is $F \sim F_{df(\text{num}), df(\text{denom})}$

Where $df(\text{num}) = df_{\text{between}}$ and $df(\text{denom}) = df_{\text{within}}$

The mean for the F distribution is $\mu = \frac{df(\text{num})}{df(\text{denom}) - 2}$

Calculating SST (total sum of squares)



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Calculating SS_{within} and SS_{between}



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Hypothesis Test with F-Statistic



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Check Your Understanding: The F Distribution and the F-Ratio





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Facts About the F Distribution

1. The curve is not symmetrical but skewed to the right.
2. There is a different curve for each set of degrees of freedom.
3. The F statistic is greater than or equal to zero.
4. As the degrees of freedom for the numerator and for the denominator get larger, the curve approximates the normal as can be seen in Figure 18. Remember that the F cannot ever be less than zero, so the distribution does not have a tail that goes to infinity on the left as the normal distribution does.
5. Other uses for the F distribution include comparing two variances and two-way Analysis of Variances. Two-Way Analysis is beyond the scope of this section.

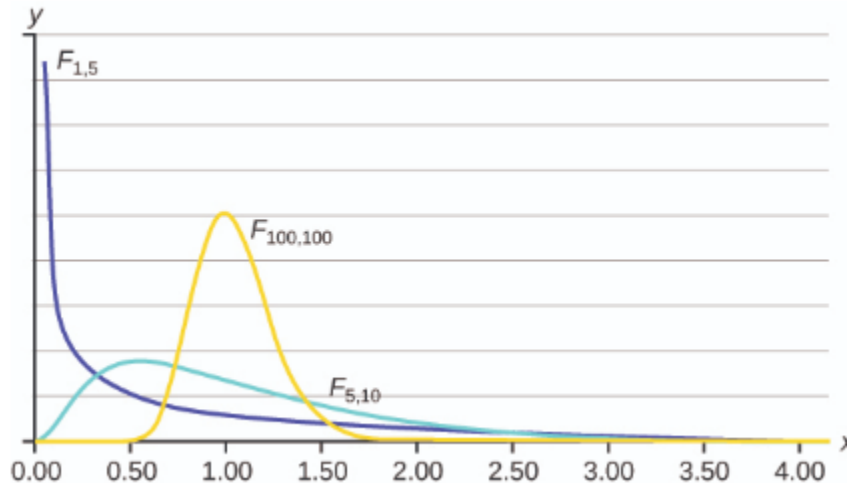


Figure 18

COMPUTE AND INTERPRET SIMPLE LINEAR REGRESSION BETWEEN TWO VARIABLES

LINEAR REGRESSION AND CORRELATION

Professionals often want to know how two or more numeric variables are related. For example, is there a relationship between the grade on the second math exam a student takes and the grade on the final exam? If there is a relationship, what is the relationship and how strong is it?

This example may or may not be tied to a model, meaning that some theory suggested that a relationship exists. This link between a cause and an effect is the foundation of the scientific method and is the core of how we determine what we believe about how the world works. Beginning with a theory and developing a model of the theoretical relationship should result in a prediction, what we have called a hypothesis earlier. Now the hypothesis concerns a full set of relationships.

The foundation of all model building is perhaps the arrogant statement that we know what caused the result we see. This is embodied in the simple mathematical statement of the functional form that $y = f(x)$. The response, Y, is caused by the stimulus, X. Every model will eventually come to this final place, and it will be here that the theory will live or die. Will the data support this hypothesis? If so, then fine, we shall believe this version of the world until a better theory comes to replace it. This is the process by which we moved from flat earth to round earth, from earth-center solar system to sun-center solar system, and on and on.

In this section we will begin with correlation, the investigation of relationships among variables that may or may not be founded on a cause-and-effect model. The variables simply move in the same, or opposite, direction. That is to say, they do not move randomly. Correlation provides a measure of the degree to which this is true. From there we develop a tool to measure cause and effect relationships, regression analysis. We will be able to formulate models and tests to determine if they are statistically sound. If they are found to be so, then we can use them to make predictions: if as a matter of policy, we changed the value of this variable what would happen to this other variable? If we imposed a gasoline tax of 50 cents per gallon how would that effect the carbon emissions, sales of Hummers/Hybrids, use of mass transit, etc.? The ability to provide answers to these types of questions is the value of regression as both a tool to help us understand our world and to make thoughtful policy decisions.

THE CORRELATION COEFFICIENT R

As we begin this section, we note that the type of data we will be working with has changed. Perhaps unnoticed, all the data we have been using is for a single variable. It may be from two samples, but it is still a univariate variable. The type of data described for any model of cause and effect is **bivariate** data — “bi” for two variables. In reality, statisticians use **multivariate** data, meaning many variables.

Data can be classified into three broad categories: time series data, cross-section data, and panel data. Time series data measures a single unit of observation; say a person, or a company or a country, as time passes. What are measures will be at least two characteristics, say the person’s income, the quantity of a particular good they buy and the price they paid. This would be three pieces of information in one time period, say 1985. If we followed that person across time we would have those same pieces of information for 1985, 1986, 1987, etc. This would constitute a time series data set.

A second type of data set is for cross-section data. Here the variation is not across time for a single unit of observation, but across units of observation during one point in time. For a

particular period of time, we would gather the price paid, amount purchased, and income of many individual people.

A third type of data set is panel data. Here a panel of units of observation is followed across time. If we take our example from above, we might follow 500 people, the unit of observation, through time, ten years, and observe their income, price paid and quantity of the good purchased. If we had 500 people and data for ten years for price, income and quantity purchased we would have 15,000 pieces of information. These types of data sets are very expensive to construct and maintain. They do, however, provide a tremendous amount of information that can be used to answer very important questions. As an example, what is the effect on the labor force participation rate of women as their family of origin, mother and father, age? Or are there differential effects on health outcomes depending upon the age at which a person started smoking? Only panel data can give answers to these and related questions because we must follow multiple people across time.

Beginning with a set of data with two independent variables we ask the question: are these related? One way to visually answer this question is to create a scatter plot of the data. We could not do that before when we were doing descriptive statistics because those data were univariate. Now we have bivariate data so we can plot in two dimensions. Three dimensions are possible on a flat piece of paper but become very hard to fully conceptualize. Of course, more than three dimensions cannot be graphed although the relationships can be measured mathematically.

To provide mathematical precision to the measurement of what we see we use the correlation coefficient. The correlation tells us something about the co-movement of two variables, but **nothing** about why this movement occurred. Formally, correlation analysis assumes that both variables being analyzed are **independent** variables. This means that neither one causes the movement in the other. Further, it means that neither variable is dependent on the other, or for that matter, on any other variable. Even with these limitations, correlation analysis can yield some interesting results.

The correlation coefficient, ρ (pronounced rho), is the mathematical statistic for a population that provides us with a measurement of the strength of a linear relationship between the two variables. For a sample of data, the statistic, r , developed by Karl Pearson in the early 1900s, is an estimate of the population correlation and is defined mathematically as:

$$r = \frac{\frac{1}{n-1} \sum (X_{1i} - \bar{X}_1)(X_{2i} - \bar{X}_2)}{s_{x1} s_{x2}}$$

OR

$$r = \frac{\sum x_{1i} x_{2i} - n \bar{X}_1 \bar{X}_2}{\sqrt{(\sum X_{1i}^2 - n \bar{X}_1^2)(\sum X_{2i}^2 - n \bar{X}_2^2)}}$$

Where S_{x1} and S_{x2} are the standard deviations of the two independent variables X_1 and X_2 , \bar{X}_1 and \bar{X}_2 are the sample means of the two variables, and X_{1i} and X_{2i} are the individual observations of X_1 and X_2 . The correlation coefficient r ranges in value from -1 to 1. The second equivalent formula is often used because it may be computationally easier.

As scary as these formulas look, they are really just the ratio of the covariance between the two variables and the product of their two standard deviations. That is to say, it is a measure of relative variances.

In practice all correlation and regression analysis will be provided through computer software designed for these purposes. Anything more than perhaps one-half a dozen observations creates immense computational problems. It was because of this fact that correlation, and even more so, regression, were not widely used research tools until after the advent of “computing machines.” Now the computing power required to analyze data using regression packages is deemed almost trivial by comparison to just a decade ago.

To visualize any **linear** relationship that may exist review the plot of a scatter diagrams of the standardized data. Figure 19 presents several scatter diagrams and the calculated value of r . In panels (a) and (b) notice that the data generally trend together, (a) upward and (b) downward. Panel (a) is an example of a positive correlation and panel (b) is an example of a negative correlation, or relationship. The sign of the correlation coefficient tells us if the relationship is a positive or negative (inverse) one. If all the values of X_1 and X_2 are on a straight line the correlation coefficient will be either 1 or -1 depending on whether the line has a positive or negative slope and the closer to one or negative one the stronger the relationship between the two variables. BUT ALWAYS REMEMBER THAT THE CORRELATION COEFFICIENT DOES NOT TELL US THE SLOPE.

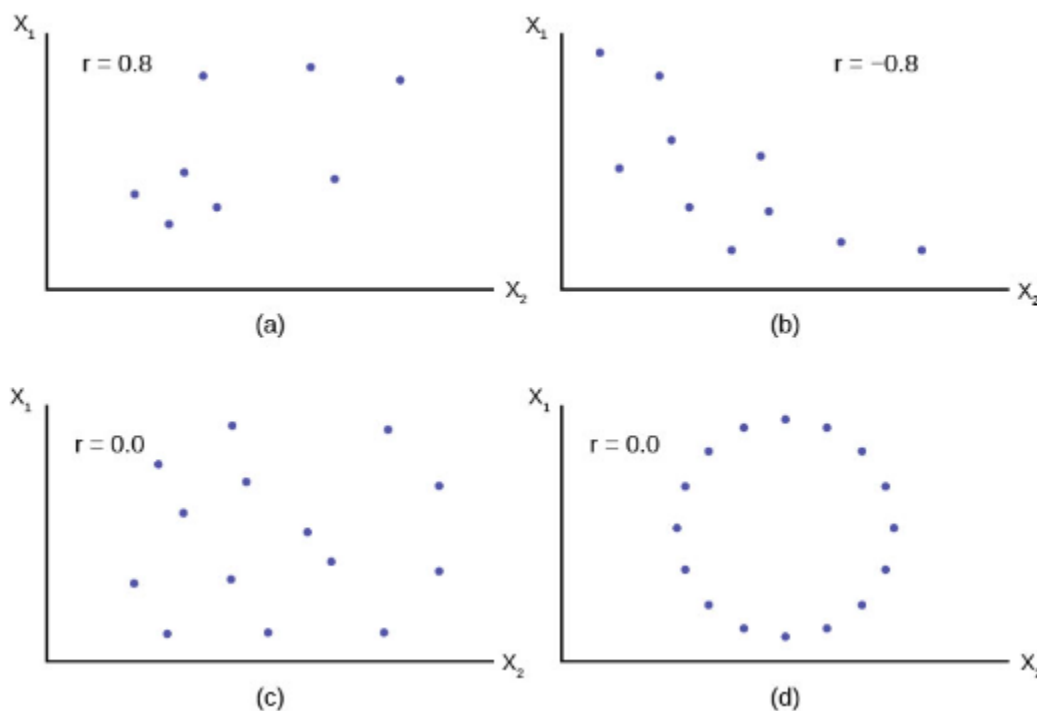


Figure 19

Remember, all the correlation coefficient tells us is whether or not the data are linearly related. In panel (d) the variables obviously have some type of very specific relationship to each other, but the correlation coefficient is zero, indicating no **linear** relationship exists.

If you suspect a linear relationship between X_1 and X_2 then r can measure how strong the linear relationship is.

What the VALUE of r tells us:

- The value of r is always between -1 and $+1$: $-1 \leq r \leq 1$.
- The size of the correlation r indicates that strength of the **linear** relationship between X_1 and X_2 . Values of r close to -1 or $+1$ indicate a stronger linear relationship between X_1 and X_2 .
- If $r = 0$ there is absolutely no linear relationship between X_1 and X_2 (**no linear correlation**).
- If $r = 1$, there is perfect positive correlation. If $r = -1$, there is perfect negative correlation. In both these cases, all of the original data points lie on a straight line: ANY straight line no matter what the slope. Of course, in the real world, this will not generally happen.

What the SIGN of r tells us

- A positive value of r means that when X_1 increases, X_2 tends to increase and when X_1 decreases, X_2 tends to decrease (**positive correlation**).
- A negative value of r means that when X_1 increases, X_2 tends to decrease and when X_1 decreases, X_2 tends to increase (**negative correlation**).

Note: Strong correlation does not suggest that X_1 causes X_2 or X_2 causes X_1 . We say, “correlation does not imply causation.”

Bivariate Relationship Linearity, Strength, and Direction



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Check Your Understanding: The Correlation Coefficient r



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Calculating Correlation Coefficient r



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Example: Correlation Coefficient Intuition



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LINEAR EQUATIONS

Linear regression to two variables is based on a linear equation with one independent variable. The equation has the form:

$$y = a + bx$$

Where a and b are constant numbers.

The variable **x is the independent variable, and y is the dependent variable**. Another way to think about this equation is a statement of cause and effect. The X variable is the cause, and the Y variable is the hypothesized effect. Typically, you choose a value to substitute for the independent variable and then solve for the dependent variable.

The graph of a linear equation of the form $y = a + bx$ is a **straight line**. Any line that is not vertical can be described by this equation.

Slope and Y-Intercept of a Linear Equation

For the linear equation $y = a + bx$, b = slope and a = y-intercept. From algebra recall that the slope is a number that described the steepness of a line, and the y-intercept is the y coordinate of

the point $(0, a)$ where the line crosses the y-axis. From calculus the slope is the first derivative of the function. For a linear function, the slope is $\frac{dy}{dx} = b$ where we can read the mathematical expression as “the change in y (dy) that results from a change in x (dx) $= b * dx$ ”

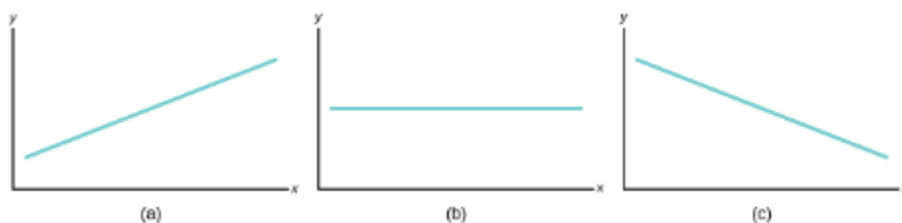


Figure 20: Three possible graphs of $y = a + bx$. (a) if $b > 0$, the line slopes upward to the right. (b) if $b = 0$, the line is horizontal. (c) if $b < 0$, the line slopes downward to the right.

THE REGRESSION EQUATION

Regression analysis is a statistical technique that can test the hypothesis that a variable is dependent upon one or more other variables. Further, regression analysis can provide an estimate of the magnitude of the impact of a change in one variable on another. This last feature, of course, is all important in predicting future values.

Regression analysis is based upon a functional relationship among variables and further, assumes that the relationship is linear. This linearity assumption is required because, for the most part, the theoretical statistical properties of non-linear estimation are not well worked out yet by the mathematicians and econometricians. This presents us with some difficulties in economic analysis because many of our theoretical models are nonlinear. The marginal cost curve, for example, is decidedly nonlinear as is the total cost function, if we are to believe in the effect of specialization of labor and the Law of Diminishing Marginal Product. There are techniques for overcoming some of these difficulties, exponential and logarithmic transformation of the data for example, but at the outset we must recognize that standard ordinary least squares (OLS) regression analysis will always use a linear function to estimate what might be a nonlinear relationship.

The general linear regression model can be stated by the equation:

$$y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \dots + \beta_k X_{ki} + \varepsilon_i$$

Where β_0 is the intercept, β_i' 's are the slope between Y and the appropriate X_i , and ε (pronounced epsilon), is the error term that captures errors in measurement of Y and the effect on Y of any variables missing from the equation that would contribute to explaining variations in Y. This equation is the theoretical population equation and therefore uses Greek letters. The equation we will estimate will have the Roman equivalent symbols. This is parallel to how we kept track of the population parameters and sample parameters before. The symbol for the population mean was μ and for the sample mean \bar{X} and for the population standard deviation was σ and for the sample standard deviation was s . The equation that will be estimated with a sample of data for two independent variables will thus be:

$$y = b_0 + b_1x_{1i} + b_2x_{2i} + e_i$$

As with our earlier work with probability distributions, this model works only if certain assumptions hold. These are that the Y is normally distributed, the errors are also normally distributed with a mean of zero and a constant standard deviation, and that the error terms are independent of the size of X and independent of each other.

Assumptions of the Ordinary Least Squares Regression Model

Each of these assumptions needs a bit more explanation. If one of these assumptions fails to be true, then it will have an effect on the quality of the estimates. Some of the failures of these assumptions can be fixed while others result in estimates that quite simply provide no insight into the questions the model is trying to answer or worse, give biased estimates.

1. The independent variables, x_i , are all measured without error, and are fixed numbers that are independent of the error term. This assumption is saying in effect that Y is deterministic, the result of a fixed component “X” and a random error component “ ε .”
2. The error term is a random variable with a mean of zero and a constant variance. The meaning of this is that the variances of the independent variables are independent of the value of the variable. Consider the relationship between personal income and the quantity of a good purchased as an example of a case where the variance is dependent upon the value of the independent variable, income. It is plausible that as income increases the variation around the amount purchased will also increase simply because of the flexibility provided with higher levels of income. The assumption is for constant variance with respect to the magnitude of the independent variable called homoscedasticity. If the assumption fails, then it is called heteroscedasticity. Figure 21 shows the case of homoscedasticity where all three distributions have the same variance around the predicted value of Y regardless of the magnitude of X.
3. Error terms should be normally distributed. This can be seen in Figure 21 by the shape of the distributions placed on the predicted line at the expected value of the relevant value of Y.
4. The independent variables are independent of Y but are also assumed to be independent of the other X variables. The model is designed to estimate the effects of independent variables on some dependent variable in accordance with a proposed theory. The case where some or more of the independent variables are correlated is not unusual. There may be no cause and effect relationship among the independent variables, but nevertheless they move together. Take the case of a simple supply curve where quantity supplied is theoretically related to the price of the product and the prices of inputs. There may be multiple inputs that may over time move together from general inflationary pressure. The input prices will therefore violate this assumption of regression analysis. This condition is called multicollinearity.
5. The error terms are uncorrelated with each other. This situation arises from an effect on one error term from another error term. While not exclusively a time series problem, it is here that we most often see this case. An X variable in time period one

has an effect on the Y variable, but this effect then has an effect in the next time period. This effect gives rise to a relationship among the error terms. This case is called autocorrelation, “self-correlated.” The error terms are now not independent of each other, but rather have their own effect on subsequent error terms.

Figure 21 shows the case where the assumptions of the regression model are being satisfied. The estimated line is $\hat{y} = a + bx$. Three values of X are shown. A normal distribution is placed at each point where X equals the estimated line and the associated error at each value of Y. Notice that the three distributions are normally distributed around the point on the line, and further, the variation, variance, around the predicted value is constant indicating homoscedasticity from assumption 2. Figure 21 does not show all the assumptions of the regression model, but it helps visualize these important ones.

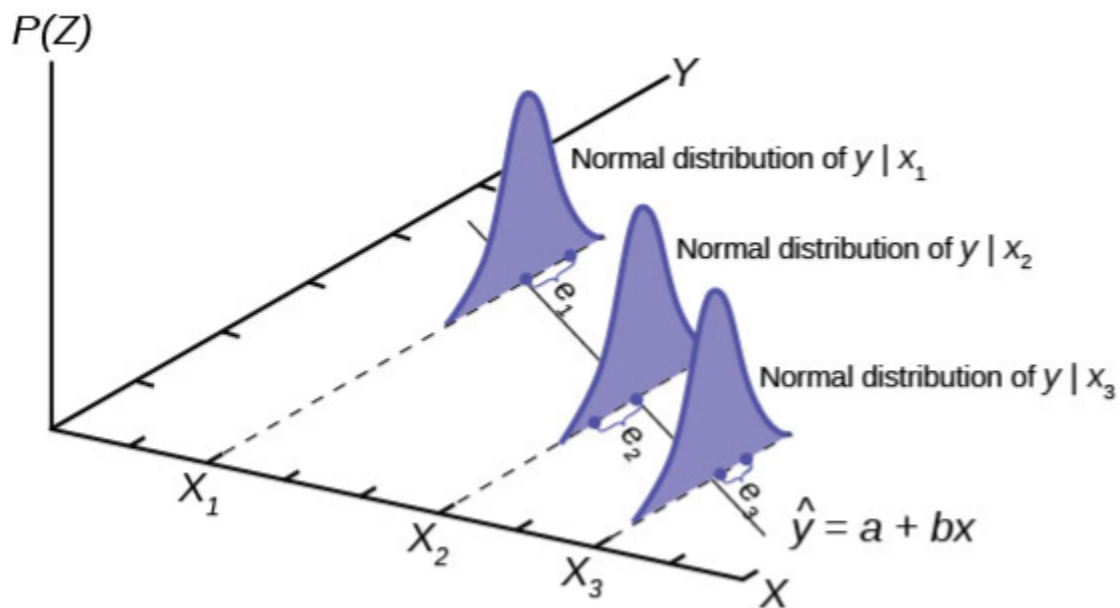


Figure 21

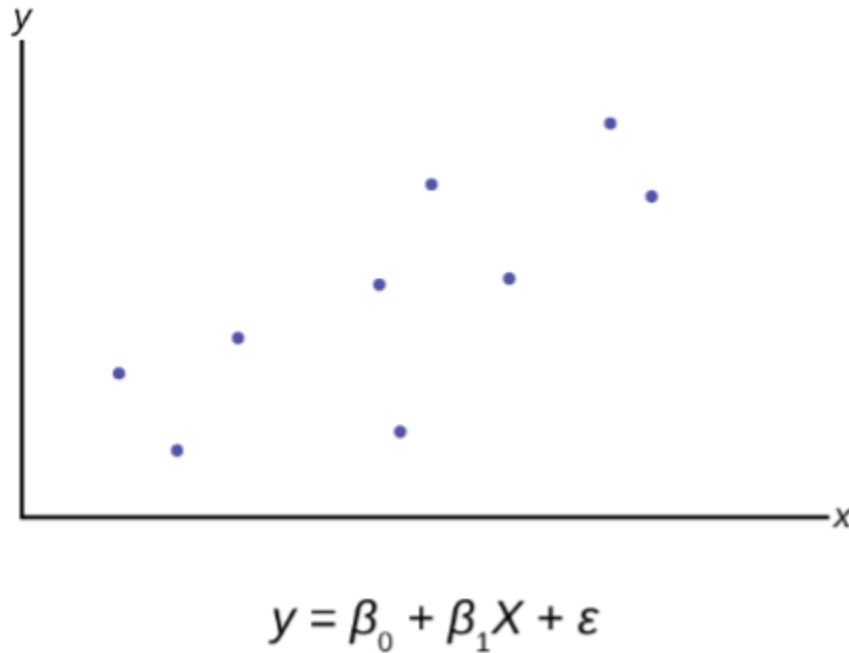


Figure 22

This is the general form that is most often called the multiple regression model. So-called “simple” regression analysis has only one independent (right-hand) variable rather than many independent variables. Simple regression is just a special case of multiple regression. There is some value in beginning with simple regression: it is easy to graph in two dimensions, difficult to graph in three dimensions, and impossible to graph in more than three dimensions. Consequently, our graphs will be for the simple regression case. Figure 22 presents the regression problem in the form of a scatter plot graph of the data set where it is hypothesized that Y is dependent upon the single independent variable X.

The regression problem comes down to determining which straight line would best represent the data in Figure 23. Regression analysis is sometimes called “least squares” analysis because the method of determining which line best “fits” the data is to minimize the sum of the squared residuals of a line put through the data.



Figure 23: Population Equation: $C = \beta_0 + \beta_1 \text{ Income} + \varepsilon$
 Estimated Equation: $C = b_0 + b_1 \text{ Income} + e$

Figure 23 shows the assumed relationship between consumption and income from macroeconomic theory. Here the data are plotted as a scatter plot and an estimated straight line has been drawn. From this graph we can see an error term, e_1 . Each data point also has an error term. Again, the error term is put into the equation to capture effects on consumption that are not caused by income changes. Such other effects might be a person's savings or wealth, or periods of unemployment. We will see how by minimizing the sum of these errors we can get an estimate for the slope and intercept of this line.

Consider the graph in Figure 24. The notation has returned to that for the more general model rather than the specific case of the Macroeconomic consumption function in our example.

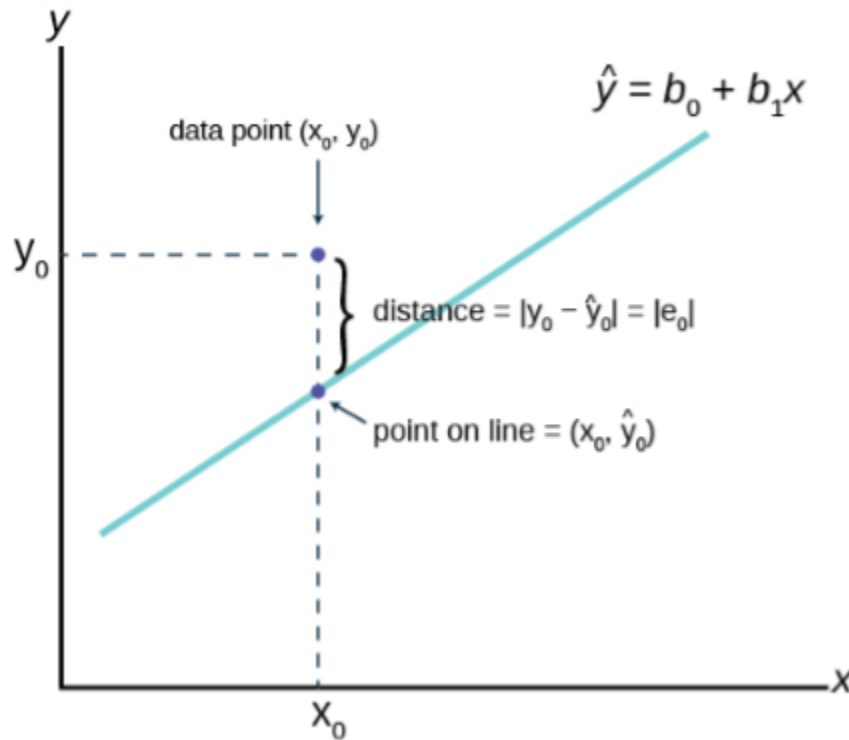


Figure 24

The \hat{y} is read “**y hat**” and is the **estimated value of y**. (In Figure 23 \hat{C} represents the estimated value of consumption because it is on the estimated line.) It is the value of y obtained using the regression line. \hat{y} is not generally equal to y from the data.

The term $y_0 - \hat{y}_0 = e_0$ is called the “**error**” or **residual**. It is not an error in the sense of a mistake. The error term was put into the estimating equation to capture missing variables and error in measurement that may have occurred in the dependent variables. The **absolute value of a residual** measure the vertical distance between the actual value of y and the estimated value of y. In other words, it measures the vertical distance between the actual data point and the predicted point on the line as can be seen on the graph at point X_0 .

If the observed data point lies above the line, the residual is positive, and the line underestimates the actual data value for y.

If the observed data point lies below the line, the residual is negative, and the line overestimates that actual data value for y.

In the graph, $y_0 - \hat{y}_0 = e_0$ is the residual for the point shown. Here the point lies above the line, and the residual is positive. For each data point the residuals, or errors, are calculated $y_i - \hat{y}_i = e_i$ for $i = 1, 2, 3, \dots, n$ where n is the sample size. Each $|e_i|$ is a vertical distance.

The sum of the errors squared is the term obviously called **Sum of Squared Errors (SSE)**.

Using calculus, you can determine the straight line that has the parameter values of b_0 and b_1

that minimizes the **SSE**. When you make the **SSE** a minimum, you have determined the points that are on the line of best fit. It turns out that the line of best fit has the equation:

$$\hat{y} = b_0 + b_1x$$

Where $b_0 = \bar{y} - b_1\bar{x}$ and $b_1 = \frac{\sum(x - \bar{x})(y - \bar{y})}{\sum(x - \bar{x})^2} = \frac{\text{cov}(x, y)}{s_x^2}$

The sample means of the x values and the y values are \bar{x} and \bar{y} , respectively. The best fit line always passes through the point (\bar{x}, \bar{y}) called the points of means.

The slope b can also be written as:

$$b_1 = r_{y,x} \left(\frac{s_y}{s_x} \right)$$

Where s_y = the standard deviation of the y values and s_x = the standard deviation of the x values and r is the correlation coefficient between x and y.

These equations are called the Normal Equations and come from another very important mathematical finding called the Gauss-Markov Theorem without which we could not do regression analysis. The Gauss-Markov Theorem tells us that the estimates we get from using the ordinary least squares (OLS) regression method will result in estimates that have some very important properties. In the Gauss-Markov Theorem it was proved that a least squares line is BLUE, which is, Best, Linear, Unbiased, Estimator. Best is the statistical property that an estimator is the one with the minimum variance. Linear refers to the property of the type of line being estimated. An unbiased estimator is one whose estimating function has an expected mean equal to the mean of the population. (You will remember that the expected value of $\frac{\mu - x}{x}$ was equal to the population mean μ in accordance with the Central Limit Theorem. This is exactly the same concept here).

Using the OLS method, we can now find the **estimate of the error variance**, which is the variance of the squared errors, e^2 . This is sometimes called the **standard error of the estimate**. (Grammatically this is probably best said as the estimate of the **error's** variance). The formula for the estimate of the error variance is:

$$s_e^2 = \frac{\sum(y_i - \hat{y}_i)^2}{n-k} = \frac{\sum e_i^2}{n-k}$$

Where \hat{y} is the predicted value of y and y is the observed value, and thus the term $(y_i - \hat{y}_i)^2$ is the squared errors that are to be minimized to find the estimates of the regression line parameters. This is really just the variance of the error terms and follows our regular variance formula. One important note is that here we are dividing by $(n - k)$, which is the degrees of freedom. The degrees of freedom of a regression equation will be the number of observations, n, reduced by the number of estimated parameters, which includes the intercept as a parameter.

The variance of the errors is fundamental in testing hypotheses for a regression. It tells us just how “tight” the dispersion is about the line. As we will see shortly, the greater the dispersion about the line, meaning the larger the variance of the errors, the less probable that the

hypothesized independent variable will be found to have a significant effect on the dependent variable. In short, the theory being tested will more likely fail if the variance of the error term is high. Upon reflection this should not be a surprise. As we tested hypotheses about a mean we observed that large variances reduced the calculated test statistic and thus it failed to reach the tail of the distribution. In those cases, the null hypotheses could not be rejected. If we cannot reject the null hypothesis in a regression problem, we must conclude that the hypothesized independent variable has no effect on the dependent variable.

A way to visualize this concept is to draw two scatter plots of x and y data along a predetermined line. The first will have little variance of the errors, meaning that all the data points will move close to the line. Now do the same except the data points will have a large estimate of the error variance, meaning that the data points are scattered widely along the line. Clearly the confidence about a relationship between x and y is affected by this difference between the estimate of the error variance.

Introduction to Residuals and Least-Squares Regression



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Calculating Residual Example



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Check Your Understanding: Linear Equations



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Residual Plots



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Check Your Understanding: Residual Plots



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Calculating the Equation of a Regression Line



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Check Your Understanding: Calculating the Equation of a Regression Line



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Interpreting Slope of Regression Line



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Interpreting y-intercept in Regression Model



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Check Your Understanding: Interpreting Slope of Regression Line and Interpreting y-intercept in Regression Model



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Using Least Squares Regression Output



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Check Your Understanding: Using Least Squares Regression Output



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How Good is the Equation?

The **multiple correlation coefficient**, also called the **coefficient of multiple determination** or the **coefficient of determination**, is given by the formula:

$$R^2 = \frac{SSR}{SST}$$

Where SSR is the regression sum of squares, the squared deviation of the predicted value of y from the mean value of y ($\hat{y} - \bar{y}$) and SST is the total sum of squares which is the total squared deviation of the dependent variable, y , from its mean value, including the error term, SSE, the sum of squared errors. Figure 25 shows how the total deviation of the dependent variable, y , is partitioned into these two pieces.

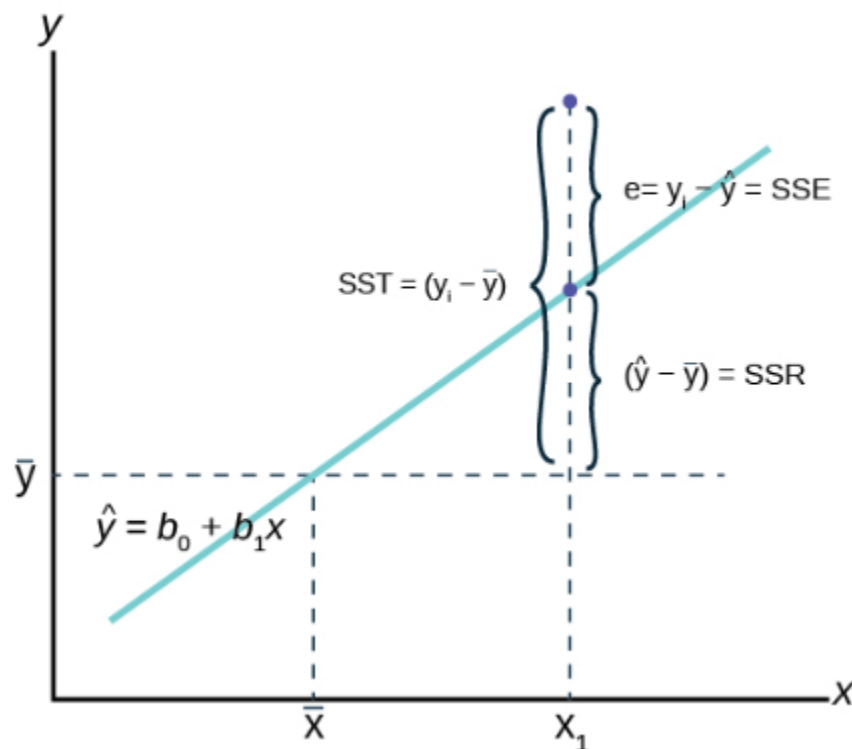


Figure 25

Figure 25 shows the estimated regression line and a single observation, x_1 . Regression analysis tries to explain the variation of the data about the mean value of the dependent variable, y . The question is, why do the observations of y vary from the average level of y ?

The value of y at observation x_1 varies from the mean of y by the difference $(y_i - \bar{y})$. The sum of these differences squared is SST, the sum of squares total. The actual value of y at x_1 deviates from the estimated value, \hat{y} , by the difference between the estimate value and the actual value, $(y_i - \hat{y})$. We recall that this is the error term, e , and the sum of these errors is SSE, sum of squared errors. The deviation of the predicted value of y , \hat{y} , from the mean value of y is $(\hat{y} - \bar{y})$ and is the SSR, sum of squares regression. It is called “regression” because it is the deviation explained by the regression. (Sometimes the SSR is called SSM for sum of squares mean because it measures the deviation from the mean value of the dependent variable, y , as shown on the graph.).

Because the $SST = SSR + SSE$ we see that the multiple correlation coefficient is the percent of the variance, or deviation in y from its mean value, that is explained by the equation when taken as a whole. R^2 will vary between zero and 1, with zero indicating that none of the variation in y was explained by the equation and a value of 1 indicating that 100% of the variation in y was explained by the equation. For time series studies expect a high R^2 and for cross-section data expect low R^2 .

While a high R^2 is desirable, remember that it is the tests of the hypothesis concerning the existence of a relationship between a set of independent variables and a particular dependent variable that was the motivating factor in using the regression model. It is validating a cause-and-effect relationship developed by some theory that is the true reason that we chose the regression analysis. Increasing the number of independent variables will have the effect of increasing R^2 . To account for this effect the proper measure of the coefficient of determination is \bar{R}^2 , adjusted for degrees of freedom, to keep down mindless addition of independent variables.

There is no statistical test for the R^2 and thus little can be said about the model using R^2 with our characteristic confidence level. Two models that have the same size of SSE, that is sum of squared errors, may have very different R^2 if the competing models have different SST, total sum of squared deviations. The goodness of fit of the two models is the same; they both have the same sum of squares unexplained, errors squared, but because of the larger total sum of squares on one of the models the R^2 differs. Again, the real value of regression as a tool is to examine hypotheses developed from a model that predicts certain relationships among the variables. These are tests of hypotheses on the coefficients of the model and not a game of maximizing R^2 .

R-Squared or Coefficient of Determination



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DATA ANALYSIS TOOLS (SPREADSHEETS AND BASIC PROGRAMMING)

DESCRIPTIVE STATISTICS

Using Microsoft Excel's "Descriptive Statistics" Tool



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How to Run Descriptive Statistics in R



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Descriptive Statistics in R Part 2



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REGRESSION ANALYSIS

How to Use Microsoft Excel for Regression Analysis

Please read this text on how to use Microsoft Excel for regression analysis.

Simple Linear Regression in Excel



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RELEVANCE TO TRANSPORTATION ENGINEERING COURSEWORK

This section explains the relevance of the regression models for trip generation, mode choice, traffic flow-speed-density relationship, traffic safety, and appropriate sample size for spot speed study to transportation engineering coursework.

REGRESSION MODELS FOR TRIP GENERATION

The trip generation step is the first of the four-step process for estimating travel demand for infrastructure planning. It involves estimating the number of trips made to and from each traffic analysis zone (TAZ). Trip generation models are estimated based on land use and trip-making data. They use either linear regression or cross-tabulation of household characteristics. Simple linear regression is described in the section above titled “Compute and Interpret Simple Linear Regression Between Two Variables”, and the tools to conduct the linear regression are discussed in “Data Analysis Tools (Spreadsheets and Basic Programming)”.

MODE CHOICE

Estimation of Mode Choice is also part of the four-step process for estimating travel demand. It entails estimating the trip makers’ transportation mode (drive alone, walk, take public transit, etc.) choices. The results of this step are the counts of trips categorized by mode. The most popular mode choice model is the discrete choice, multinomial logit model. Hypothesis tests are conducted for the estimated model parameters to assess whether they are “statistically significant.” The section titled “Use Specific Significance Tests Including, Z-Test, T-Test (one and two samples), Chi-Squared Test” of this chapter provides extensive information on hypothesis testing.

TRAFFIC FLOW-SPEED-DENSITY RELATIONSHIP

Greenshield’s model is used to represent the traffic flow-speed-density relationship. Traffic speed and traffic density (number of vehicles per unit mile) are collected to estimate a linear regression model for speed as a function of density. “Compute and Interpret Simple Linear Regression Between Two Variables” above provides information on simple linear regression. “Data Analysis Tools (Spreadsheets and Basic Programming)” provides guidance for implementing the linear regression technique using tools available in Microsoft Excel and the programming language R.

TRAFFIC SAFETY

Statistical analyses of traffic collisions are used to estimate traffic safety in roadway locations. A variety of regression models, most of which are beyond the scope of this book, are implemented to investigate the association between crashes and characteristics of the roadway locations. Hypothesis tests are conducted for the estimated model parameters to assess whether they are “statistically significant.” “Find Confidence Intervals for Parameter Estimates” above describes the confidence intervals of the parameters, and “Use Specific Significance Tests Including, Z-Test, T-Test (one and two samples), Chi-Squared Test” discusses the different types of hypothesis tests. Programming language R includes statistical analysis toolkits or packages that may be used to estimate the regression models for crash data.

APPROPRIATE SAMPLE SIZE FOR SPOT SPEED STUDY

Spot speed studies during relatively congestion-free durations are conducted to estimate mean speed, modal speed, pace, standard deviation, and different percentile of speeds at a roadway location. An adequate number of data points (i.e., the number of vehicle speeds recorded) are required to ensure reliable results within an acceptable margin of error. “Estimate the Required Sample Size for Testing” in this chapter discusses the approach to assessing the adequacy of sample size.

Key Takeaways

- Estimating the total number of trips made to and from each traffic analysis zone (TAZ) is typically the first step in the four-step travel demand modeling process. Simple linear regression models and the tools (such as MS Excel and R) to estimate these models are used this step of the travel demand modeling process. The independent variables for these models are the total number of trips with demographic and employment data for the zones as independent variables.
- Hypothesis testing is used to determine the statistical significance of coefficients estimated for the discrete choice multinomial models. Such models are used for the mode choice step of the travel demand modeling process.
- Linear regression models are also used to estimate the relationship between speed and density on uninterrupted flow facilities such as freeway segments.
- Regression models are also implemented to investigate the association between crashes and characteristics of the roadway locations. Tools described in the chapter for linear regression may also be used for estimating these models, including negative binomial regression models.
- Statistical analysis is also used to ensure that an adequate number of data points (i.e., the number of vehicle speeds recorded) are used to obtain estimates of the mean, median, and 85th percentile of speed within an acceptable margin of error in a spot speed study.

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DERIVATIVE NOTES

Minor revisions include editing language for clarity, length, and flow as well as corrections to and additions of hyperlinks and citations.

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